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Hardy fields in several variables

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Algebra. — *Hardy fields in several variables.* Nota (*) di LEONARDO PASINI, presentata dal Socio G. ZAPPA.

RIASSUNTO. — In questo lavoro si estende il concetto di campo di Hardy [Bou], al contesto dei germi di funzioni in più variabili che sono definite su insiemi semi-algebrici [Br.], [D.] e che risultano essere morfismi di categorie lisce [Pal.]. In tale contesto si dimostra che per ogni campo di Hardy di germi di una fissata categoria liscia $\mathscr{C}$, la sua chiusura algebrica relativa nell'anello G $\mathscr{C}$, di tutti i germi nella stessa categoria liscia, è un campo di Hardy reale chiuso, che è l'unica chiusura reale del campo di Hardy fissato nell'anello G $\mathscr{C}$ dei germi di funzioni continue.

Viene quindi generalizzato un risultato di Robinson [R.], inerente i campi di Hardy tradizionali su R rispetto alla categoria C^∞ e dove i germi sono presi a $+\infty$. Inoltre, per ogni categoria liscia $\mathscr{C}$ si definisce la classe dei « $\mathscr{C}$–campi » e si nota che lo stesso risultato sulla chiusura reale è valido in questo caso.

I. Utilising Palais' notion of a smoothness category [Pal.], indicated by $\mathscr{C}$ a category whose objects are open subsets θ of finite dimension real vector spaces and whose morphisms $\mathscr{C}(\theta_1, \theta_2)$ are certain C^1 functions from θ_1 to θ_2 with the usual composition law as composition law, we define:

DEFINITION 1.1. *A category $\mathscr{C}$ is said to be a smoothness category if the following conditions are satisfied:*

1) *if θ is an object of $\mathscr{C}$ and* V *is a finite dimension real vector space then $\mathscr{C}(\theta, V)$ is a linear subspace of the real vector space $C^1(\theta, V)$ of all C^1 functions from θ to* V *and contains all costant functions from θ to* V.

2) *if $V_1, \ldots, V_m$ and* W *are finite dimension real vector spaces then $\mathscr{C}(V_1 + \ldots + V_m, W)$ contains all multilinear functions.*

3) *let θ_1 and θ_2 be open subsets, respectively, of the finite dimension real vector spaces V_1 and V_2. A function $f\colon \theta_1 \to \theta_2$ is in $\mathscr{C}(\theta_1, \theta_2)$ if for any $x \in \theta_1$ there is an open subset $\theta_x \subseteq \theta_1$, containing x such that $f_{|\theta x} \in \mathscr{C}(\theta_x, V_2)$.*

4) *if $f_1 \in \mathscr{C}(\theta, V_1)$ and $f_2 \in \mathscr{C}(\theta, V_2)$ then $x \mapsto (f_1(x), f_2(x))$ is in $\mathscr{C}$ $(\theta, V_1 \times V_2)$.*

5) *if $f \in \mathscr{C}(\theta_1, \theta_2)$ is a bijection from θ_1 to θ_2 then $f^{-1} \in \mathscr{C}(\theta_2, \theta_1)$ if f^{-1} is C^1 (or equivalently if Df_x is non-singular for any $x \in \theta_1$).*

(*) Pervenuta all'Accademia il 9 ottobre 1985.

From the definition we can deduce immediately, that $\mathscr{C}(\theta, \mathbf{R})$ is a ring with the pointwise defined operations. Moreover, for any smoothness category $\mathscr{C}$, it is possible to prove that the implicit function theorem is valid.

Examples of smoothness categories are: the categories $\mathscr{C}^k$ $(k = 1, \ldots, \infty)$ of (whose morphisms are) C^k functions; the category $\mathscr{C}^\omega$ of analytic functions; the Hölder categories $\mathscr{C}^{k+\alpha}$ $(k \in \mathbf{N}^+,\ 0 < \alpha < 1)$; the Lipschitz categories $\mathscr{C}^{k-}$, $k \in \mathbf{N}^+$; the category $\mathscr{C}^\Omega$ of Nash functions.

2. Let 0 be a point belonging to the one point compactification $(\mathbf{R}^n)^+ = \mathrm{R}^n \cup \{\alpha\}$ $(n \in \mathbf{N}^+,\ \alpha \notin \mathbf{R})$ of the euclidean space $\mathbf{R}^n$.

Moreover, let A be a semi-algebraic subset of $\mathbf{R}^n$ such that 0 belongs to the closure $\bar{\mathrm{A}}$ of A in $(\mathbf{R}^n)^+$ and there exists $\varepsilon_0 \in \mathbf{R}^+$ for which, for each $\varepsilon < \varepsilon_0$, $\varepsilon \in \mathbf{R}^+$, the set $\mathrm{I}(0, \varepsilon) \cap \mathrm{A}$, where

$$\mathrm{I}(0,\varepsilon) = \begin{cases} \{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid \sum_{i=1}^{n} (x_i - 0_i)^2 < \varepsilon^2\} & \text{if } 0 = (0_1, \ldots, 0_n) \in \mathbf{R}^n \\ \{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid \sum_{i=1}^{n} x_i^2 > 1/\varepsilon^2\} & \text{if } 0 = \alpha \notin \mathbf{R}^n \end{cases}$$

s a connected subset of $\mathbf{R}^n$.

We denote by $\mathrm{H}_{\mathrm{A}_0}$ the set of real valued functions defined on a subset of $\mathbf{R}^n$ such that there exists $\varepsilon \in \mathbf{R}^+$ and a neighbourhood N of $\mathrm{I}(0, \varepsilon) \cap \mathrm{A}$ for which $\mathrm{N} \subseteq \mathrm{Dom}\, f$. We introduce an equivalence relation $\sim$ over $\mathrm{H}_{\mathrm{A}_0}$ in the following manner:

$f \sim g$ if there exists $\varepsilon \in \mathbf{R}^+$ for which $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$ is valid in a neighbourhood of $\mathrm{I}(0, \varepsilon) \cap \mathrm{A}$.

$\mathrm{G}(\mathrm{H}_{\mathrm{A}_0})$, the quotient set of $\mathrm{H}_{\mathrm{A}_0}$ by the equivalence relation $\sim$ is, with the obviously defined operations, a unitary commutative ring.

Indicated by $\mathscr{C}$ any smoothness category we define:

DEFINITION 2.1. *An element* ψ *of* $\mathrm{G}(\mathrm{H}_{\mathrm{A}_0})$ *is said to be of class* $\mathscr{C}$ *if there exists* $f \in \mathrm{H}_{\mathrm{A}_0}$ *such that:*

1) $f \in \psi$,

2) $f \in \mathscr{C}(\mathrm{N}, \mathbf{R})$, *where* N *is a neighbourhood of* $\mathrm{I}(0, \varepsilon) \cap \mathrm{A}$, *for a certain* $\varepsilon \in \mathbf{R}^+$.

DEFINITION 2.2. *An element* ψ *of* $\mathrm{G}(\mathrm{H}_{\mathrm{A}_0})$ *is said to be semi-algebraic if there exists* $f \in \mathrm{H}_{\mathrm{A}_0}$ *such that:*

1) $f \in \psi$,

2) f *is semi-algebraic.*

3. We denote by $G\mathscr{C}(H_{A_0})$ and $G\mathscr{C}_{s.a}(H_{A_0})$, respectively, the subrings of $G(H_{A_0})$ formed by the elements of class $\mathscr{C}$ and by the elements semi-algebraic of class $\mathscr{C}$.

DEFINITION 3.1. *By a « $\mathscr{C}$-field in* 0 *for* A » *we mean a subring* $\exists$ *of* $G\mathscr{C}(H_{A_0})$ *such that*:

$\exists$ *is a field with the operations of* $G\mathscr{C}(H_{A_0})$.

DEFINITION 3.2. *By a « semi-algebraic $\mathscr{C}$-field in* 0 *for* A » *we mean a subring* $\exists$ *of* $G\mathscr{C}_{s.a.}(H_{A_0})$ *such that*:

$\exists$ *is a field with the operations of* $G\mathscr{C}_{s.a.}(H_{A_0})$.

DEFINITION 3.3. *By a « $\mathscr{C}$-Hardy field in n-variables in* 0 *for* A » *we mean a subring* $\exists$ *of* $G\mathscr{C}(H_{A_0})$ *such that*:

1) $\exists$ *is a field with the operations of* $G\mathscr{C}(H_{A_0})$,

2) *if* $\psi \in \exists$ *then* $\psi_i \in \exists$, *where* $\psi_i = \left[\dfrac{\partial f}{\partial x_i}\right]$, *for* $i = 1, \ldots, n$ *and* $f \in \psi$.

Particularly, the class of « semi-algebraic $\mathscr{C}$-Hardy fields in several variables in 0 for A », for any smoothness category $\mathscr{C}$, turns out to coincide with the class of « $\mathscr{C}^{\Omega}$-Hardy fields in several variables in 0 for A ».

For the proofs and details about the results we are going to state, see [Pas.].

If $\exists$ is a generic field belonging to any of the classes defined above, it is possible to prove the following results:

PROPOSITION 3.4. *The set* $P = \{\psi \in \exists \mid$ *there exists* $f \in \psi$ *and* $\varepsilon \in \mathbf{R}^+$, *such that* $f(x_1, \ldots, x_n) > 0$ *is valid on* $I(0, \varepsilon) \cap A\}$ *turns out to be a total ordering on* $\exists$.

THEOREM 3.5. *There exists a real closed field belonging to the same class of* $\exists$ *and containing* $\exists$.

In fact we prove that, for any field $\exists$ as above, there exist some of its real closures which belong to the same class of $\exists$.

Moreover we can prove a result of uniqueness for such real closures.

In fact, denoting by ${}^{D}\exists$ the relative algebraic closure of $\exists$ in D, where D is any subring of $G(H_{A_0})$ containing $\exists$ and denoting by $GC(H_{A_0})$ the subring of $G(H_{A_0})$ formed by the germs of continuous functions in 0 for A, we obtain:

THEOREM 3.6. *If* $\mathscr{M}$ *is any real closed field belonging to the same class of* $\exists$ *and* $\exists \subseteq \mathscr{M}$, *then*: ${}^{GC(H_{A_0})}\exists \subseteq \mathscr{M}$.

Supposing $\mathscr{C}$ is the smoothness category characterizing the field $\exists$, it follows:

COROLLARY 3.7. *If $\mathcal{M}$ is any of the real closures of* $\exists$ *belonging to the same class of fields of* $\exists$ *then*: $\mathcal{M} = {}^{G\mathcal{C}(H_{A_0})}\exists$.

Moreover it is easy to prove that:

COROLLARY 3.8. *If* $[y(\bar{x})]_\sim \in {}^{GC(H_{A_0})}\exists$ *then* $\exists\,[[y(\bar{x})]_\sim]$ *belongs to the same class of fields of* $\exists$.

Finally, we note that the rings of germs of real semi-algebraic functions in one variable, taken at $\pm\infty$ in the usual manner or at any real point from right or left, that is, $G_{s.a.}(H_{A_0})$ where $0 \in (\mathbf{R})^+$ and A is a real interval of the form $(0, \varepsilon)$ or $(\varepsilon, 0)$, $\varepsilon \in \mathbf{R}$, are real closed Nash Hardy fields. In fact, we know [V.d.D.] that any semi-algebraic real function is continuous at $\pm\infty$ and near any $0 \in \mathbf{R}$ from right or left. Then, choosing A as above, for any $0 \in (\mathbf{R})^+$ we have:

$G_{s.a.}(H_{A_0}) = {}^{GC(H_{A_0})}\mathbf{R}_{A_0}[x] = {}^{GC(H_{A_0})}\mathbf{R}_{A_0}(x)$ that is a real closed Nash Hardy field by corollary 3.7. This is obvious since $\mathbf{R}_{A_0}(x)$, the ring of the germs in 0 for A of rational functions, is a Nash Hardy field.

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