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Geometria algebrica. — *On the ampleness of* $K_X \otimes L^n$ *for a polarized threefold* (X , L) (*). Nota di ANTONIO LANTERI e MARINO PALLESCHI (**), presentata (***) dal Socio E. MARCHIONNA.

RIASSUNTO. — Siano X una varietà algebrica proiettiva complessa non singolare tridimensionale, L un fibrato lineare ampio su X, e $n \geq 2$ un intero. Si prova che, a meno di contrarre un numero finito di (— 1)–piani di X, il fibrato $K_X \otimes L^n$ è ampio ad eccezione di alcuni casi esplicitamente descritti. Come applicazione si dimostra l'ampiezza del divisore di ramificazione di un qualunque rivestimento di $\mathbf{P}^3$ o della quadrica liscia di $\mathbf{P}^4$.

1. INTRODUCTION

The subject of this paper is the ampleness of $K_X \otimes L^n$ on a polarized threefold, i.e. a pair (X , L) where X is a complex connected projective algebraic threefold and L an ample line bundle on X. We prove that, up to contracting a finite number of (— 1)-planes of X, $K_X \otimes L^n$ is ample if $n \geq 2$, apart from a few cases explicitly described (Theorem 2.1). This fact together with known results on surfaces [5] implies that $K_X \otimes L^{k+1}$ is ample for any polarized manifold $(X, L) \neq (\mathbf{P}^k, \mathcal{O}_{\mathbf{P}^k}(1))$ of dimension $k \leq 3$. If the same were true in every dimension, it would extend Ein's result on the ampleness of the ramification divisor of a branched covering of $\mathbf{P}^k$ [2]. Partial results are provided by Propositions 2.4, 2.5. On the other hand Ein's result can be generalized in a different perspective. Actually, as an application of Theorem 2.1, we show the ampleness of the ramification divisor of any branched covering of a quadric threefold (Theorem 3.2).

2.

Let (X , L) be a polarized threefold. As usual we shall not distinguish between line bundles and invertible sheaves. We also write L^n for $L^{\otimes n}$. Following Sommese [6] we shall call a polarized threefold (X′ , L′) a reduction of (X , L) if there is a surjective morphism $\pi : X \to X'$ such that *i*) π is the blow-up of a finite set $F \subset X'$ and ii) $\pi^* L' = L \otimes [\pi^{-1}(F)]$. K_X will stand for the canonical bundle of X.

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(2.1) THEOREM. *Let* (X, L) *be a polarized threefold. The line bundle* $K_X \otimes L^n$ *is ample for* $n \geq 2$, *apart from the following cases:*

$n = 4$ *and* $(X, L) = (\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1))$;

$n = 3$ *and either* $(X, L) = (\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1))$, $(X, L) = (Q, \mathcal{O}_Q(1))$, Q *being a smooth hyperquadric of* $\mathbf{P}^4$, *or* X *is a* $\mathbf{P}^2$*-bundle and* $L_{|F} = \mathcal{O}_{\mathbf{P}^2}(1)$ *for any fibre* F *of* X;

$n = 2$ *and either*

(*a*) X *is a* $\mathbf{P}^1$*–bundle and* $L_{|F} = \mathcal{O}_{\mathbf{P}^1}(1)$,

(*b*) X *is a* $\mathbf{P}^2$*–bundle and* $L_{|F} = \mathcal{O}_{\mathbf{P}^2}(e)$, $e = 1$ *or* 2,

(*c*) X *is a quadric bundle and* $L_{|F} = \mathcal{O}_F(1)$, *where, in each case,* F *is a fibre of* X,

(*d*) X *is a Fano threefold of index* $r \geq 2$, $\operatorname{Pic}(X) \simeq \mathbf{Z}[l]$ *and* $L = l^m$, $m < r$, *or there is a reduction* (X', L') *of* (X, L) *where* $K_{X'} \otimes L'^2$ *is ample.*

Proof. Let us consider the line bundle $N = K_X \otimes L^{n-1}$. First assume that N^s is spanned by its global sections for some $s > 0$. So by tensoring N^s with the ample line bundle L^s, we get the ampleness of $N \otimes L = K_X \otimes L^n$. Now assume that for no $s > 0$ N^s is spanned by its global sections and let $M = N \otimes K_X^{-1} = L^{n-1}$. Then since $K_X^n \otimes M^n$ is spanned for no $n > 0$, it follows from [1, Thm. 2.2] that either (X, M) is one of the pairs listed in (*a*)–(*d*) or it admits a reduction (X', M') such that some power of $K_{X'} \otimes M'$ is spanned by its global sections. In the latter case, let $\pi : X \to X'$ be the reduction morphism and let $E_1, \ldots, E_t$ be the (-1)–planes contracted by π. Since $\pi^* M' = M \otimes [E_1] \otimes \ldots \otimes [E_t]$, by restriction to E_i, we get

$$M_{|E_i} = [E_i]^{-1}{}_{|E_i} = \mathcal{O}_{\mathbf{P}^2}(1).$$

On the other hand $M = L^{n-1}$ and therefore this case can occur only when $n = 2$. It only remains to see which of the exceptions (*a*)–(*d*) are allowable for the pair (X, M) when $n \geq 3$. Since $M = L^{n-1}$, cases (*a*) and (*c*) cannot occur, whereas case (*b*) happens if and only if $M_{|F} = \mathcal{O}_{\mathbf{P}^2}(2)$, which means that $n = 3$ and (X, L) is as in (*b*) with $e = 1$. Finally assume that (X, M) is as in (*d*). Then we have

$$L^{n-1} = M = l^m, \; m < r,$$

where l is the ample generator of $\operatorname{Pic}(X)$ and r is the index of the Fano threefold X. Since $r \leq 4$ and $n \geq 3$ we have the following possibilities: $n = 4 = r$, in which case $X \simeq \mathbf{P}^3$ and $L = l$, $n = 3 \leq r \leq 4$, in which case $L = l$ and X is either $\mathbf{P}^3$ or a quadric hypersurface. q. e. d.

By summing up some known results in dimension less than 3, we get

(2.2) COROLLARY. *Let* (X, L) *be a polarized manifold of dimension* $k \leq 3$. *If* $(X, L) \not\simeq (\mathbf{P}^k, \mathcal{O}_{\mathbf{P}^k}(1))$, *then* $K_X \otimes L^n$ *is ample for any* $n \geq k+1$.

For $k=1$ this is a trivial fact; for $k=2$ see [5].

This suggests the following

(2.3) QUESTION. Is $K_X \otimes L^{k+1}$ ample for any dimension $k \geq 1$, apart from the obvious exception $(\mathbf{P}^k, \mathcal{O}_{\mathbf{P}^k}(1))$?

As is known the answer is affirmative when L is very ample. This could be deduced indirectly from the finiteness of the Gauss map [3] and Lemma 4 of [2]; but, what's more, using the standard technique of separating points and tangent vectors, one can directly prove, by induction.

(2.4) PROPOSITION. *If* L *is very ample, then* $K_X \otimes L^{k+1}$ *is very ample unless* $(X, L) \simeq (\mathbf{P}^k, \mathcal{O}_{\mathbf{P}^k}(1))$.

In a very special case (2.3) can be answered affirmatively.

(2.5) PROPOSITION. *Assume that* $\operatorname{Pic}(X) \simeq \mathbf{Z}$; *then* $K_X \otimes L^{k+1}$ *is ample unless* $(X, L) \simeq (\mathbf{P}^k, \mathcal{O}_{\mathbf{P}^k}(1))$.

Proof. Let l be the ample generator of Pic (X). Then $K_X = l^{-r}$, $r \in \mathbf{Z}$. Of course there is nothing to prove when $r \leq 0$ and so we can assume $r > 0$. This means that X is a Fano manifold of index r. Let $L = l^n$, $n > 0$ and assume that $K_X \otimes L^{k+1} = l^{n(k+1)-r}$ is not ample. This yields $n(k+1) - r \leq 0$. If equality occurs, then we get $(X, L) \simeq (\mathbf{P}^k, \mathcal{O}_{\mathbf{P}^k}(1))$, by [4], Th. 2.1. So it is enough to prove that it cannot be $r > n(k+1)$. Actually we prove that $r \leq k+1$. To see this, put $\chi(m) = \chi(l^m)$. By the Kodaira vanishing theorem we know that $h^i(l^m) = 0$ if $m < 0$ and $i = 0, \ldots, k-1$. So, by Serre's duality, $\chi(m) = (-)^k h^k(l^m) = (-)^k h^0(l^{-(l+m)})$ if $m < 0$. Therefore

$$\chi(m) = 0 \quad \text{for} \quad -r \leq m < 0 \,. \tag{2.5.1}$$

On the other hand, since X is Fano, $\chi(0) = \chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) = 1$ by the Kodaira vanishing theorem again. Hence $\chi(m)$, a polynomial of degree k which does not vanish everywhere, has at most $k+1$ distinct roots. It thus follows from (2.5.1) that $r \leq k+1$. q. e. d.

3.

Let $f: X \to Y$ be a finite morphism of projective manifolds of dimension k. The ramification formula gives $R \in |K_X \otimes f^* K_Y^{-1}|$, where R stands for the ramification divisor of f on X. Now assume that $K_Y^{-1} = N^t$, with N ample and $t > 0$ (this is equivalent to saying that Y is a Fano manifold whose index

r is divided by t). Since f is a finite morphism, the line bundle $L = f^* N$ is ample and

$$\text{(3.0)} \qquad R \in | K_X \otimes L^t | , \; L \text{ ample.}$$

Hence (2.1) applies to studying the ampleness of the ramification divisor of branched coverings of Fano manifolds.

(3.1) *Example.* Take $Y = \mathbf{P}^k$; so (3.0) becomes $R \in | K_X \otimes L^{k+1} |$. By Corollary 2.2, if $k \leq 3$ we get the ampleness of R with the trivial exception $\deg f = 1$. If the answer to Question 2.3 were affirmative, then we could obtain the ampleness of R for any k. Actually the ampleness of the ramification divisor of a branched covering of $\mathbf{P}^k$ was proved by Ein [2] answering a question asked by Lazarsfeld. This fact might be a good reason to hope that the answer to (2.3) is yes.

At least when $k \leq 3$, the above results allow us to study the ampleness of R when $\mathbf{P}^k$ (i.e. the Fano manifold of index $r = k + 1$) is replaced with a quadric (i.e. a Fano manifold of index $r = k$).

(3.2) THEOREM. *Let* $f : X \to Q$ *be a finite morphism from a manifold* X *to a smooth quadric* Q *of dimension* $k \leq 3$. *The ramification divisor* R *is ample unless either*

i) f *is an isomorphism, or*

ii) $k = 2$, X *is a* $\mathbf{P}^1$*-bundle,* $f^* \mathcal{O}_Q(1)_{|F} = \mathcal{O}_{\mathbf{P}^1}(1)$ *for any fibre* F *of* X *and* R *is a sum of fibres.*

Proof. Using the above notation, $R \in | K_X \otimes f^* \mathcal{O}_Q(1)^k |$, since $K_Q = \mathcal{O}_Q(-k)$. Then R is ample unless either

(α) $(X, L) \simeq (\mathbf{P}^k, \mathcal{O}_{\mathbf{P}^k}(1))$,

(β) $(X, L) \simeq (Q, \mathcal{O}_Q(1))$, or

(γ) X is a $\mathbf{P}^{k-1}$-bundle and $L_{|F} = \mathcal{O}_{\mathbf{P}^{k-1}}(1)$ for any fibre F of X.

This follows from Theorem 2.1 for $k = 3$ and from [5], Th. 2.5, in case $k = 2$.

Let $H \in | \mathcal{O}_Q(1) |$; then

$$(f^* \dot{H})^k = (\deg f)(\dot{H})^k = 2 \deg f.$$

But $(f^* \dot{H})^k = 1$ or 2 according to whether we are in case (α) or (β). Therefore case (α) cannot occur, whereas $\deg f = 1$ in case (β). In case (γ) let $g = f_{|F}$; since $g^* \mathcal{O}_Q(1) = \mathcal{O}_{\mathbf{P}^{k-1}}(1)$, g embeds F ($= \mathbf{P}^{k-1}$) into Q as a linear space of dimension $k - 1$. As Q is assumed to be smooth, this can only occur when $k = 2$. In this case $K_{X|F} = \mathcal{O}_{\mathbf{P}^1}(-2)$, since X is a $\mathbf{P}^1$-bundle, and then $(K \otimes L^2)_{|F} = 0$. Therefore R is a sum of fibres, since it belongs to $| K_X \otimes L^2 |$.

Added in proof. Question (2.3) has recently been given a positive answer by T. Fujita and P. Ionescu, independently.

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