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Michele Carriero, Antonio Leaci, Eduardo Pascali

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Calcolo delle variazioni. — *Integrals with respect to a Radon measure added to area type functionals : semi-continuity and relaxation* (*).

Nota di MICHELE CARRIERO (**), ANTONIO LEACI (***) e EDUARDO PASCALI (**), presentata (****) dal Corrisp. E. DE GIORGI.

RIASSUNTO. — Diamo condizioni sulle funzioni f, g e sulla misura μ affinché il funzionale

$$F(u) = \int_{\Omega} f(x, u, Du)\,dx + \int_{\overline{\Omega}} g(x, u)\,d\mu$$

sia $L^1(\Omega)$-semicontinuo inferiormente su $W^{1,1}(\Omega) \cap C^0(\overline{\Omega})$.

Affrontiamo successivamente il problema del rilassamento.

INTRODUCTION

In this Note we present some results that deal with $L^1(\Omega)$ – lower semicontinuity (l.s.c.) and with the relaxation problem for the functionals

$$F(u) = \int_{\Omega} f(x, u, Du)\,dx + \int_{\overline{\Omega}} g(x, u)\,d\mu \tag{1}$$

on $W^{1,1}(\Omega) \cap C^0(\overline{\Omega})$.

It is well known that such functionals occur in many questions of Calculus of variations (Plateau's problem ([4], [9], [10], [11], [12]) and problems of obstacles ([1], [3], [8])).

Several Authors (see [7] and quoted references) have investigated conditions that ensure the $L^1(\Omega)$ — l.s.c. of the functional

$$u \longrightarrow \int_{\Omega} f(x, u, Du)\,dx\,. \tag{2}$$

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(**) Dipartimento di Matematica, Lecce.

(***) Facoltà di Ingegneria, Potenza.

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Here, under the assumption that the functional (2) is $L^1(\Omega)$ — l.s.c., we intend to study the following problems (see also [5]):

(P 1) If μ is a Radon measure on R^n, to determine conditions on g which imply that the functional (1) is $L^1(\Omega)$ — l.s.c. on $W^{1,1}(\Omega) \cap C^0(\overline{\Omega})$.

(P 2) If the functional (1) is not $L^1(\Omega)$ — l.s.c., to find conditions on f, g and μ under which a function γ exists such that

$$\int_{\Omega} f(x, u, Du)\,dx + \int_{\overline{\Omega}} \gamma(x, u)\,d\mu$$

is the greatest $L^1(\Omega)$ — l.s.c. functional smaller than (1) on $W^{1,1}(\Omega) \cap C^0(\overline{\Omega})$ (henceforth $sc^-(L^1(\Omega))\,F$).

The scheme of the paper is as follows:

1) Measure-theoretic preliminaries.
2) Construction of a measure connected with functional (1).
3) Results on semi-continuity and relaxation.

§ 1. Let $\mathscr{F}$ be a σ-algebra of subsets of a non empty set X and let α and β be two countably subadditive set functions on $\mathscr{F}$.

We shall denote by $\alpha \wedge \beta$ the greatest countably subadditive set function smaller than, or equal to, both α and β.

It is easy to check that if α and β are measures on $\mathscr{F}$ then also $\alpha \wedge \beta$ is a measure on $\mathscr{F}$.

The following characterization also holds: for $B \in \mathscr{F}$

$$(\alpha \wedge \beta)(B) = \inf\{\alpha(B \cap L) + \beta(B \setminus L);\ L \in \mathscr{F}\}.$$

Let now α and β be measures on $\mathscr{F}$, with β finite.

PROPOSITION 1. *There exists* $\lambda : X \to [0, +\infty]$ $\mathscr{F}$*-measurable such that:*

(i) $$(\alpha \wedge \rho\,\beta)(B) = \int_B (\lambda(x) \wedge \rho)\,d\beta(x) \qquad \textit{for all } \rho > 0,\ B \in \mathscr{F};$$

(ii) $$\alpha(B) = \int_B \lambda\,d\beta$$

for every $B \in \mathscr{F}$ *such that* $B \subseteq \{\lambda < +\infty\}$.

Henceforth we shall adopt for λ the symbol $d\alpha/d\beta$.

§ 2. We now introduce a new measure associated with functional (1); this generalizes De Giorgi's geometric measure (see [4], [6], [2], [12]).

Let $\psi : \mathrm{R}^n \longrightarrow \mathrm{R}$ be a function such that

$$(3)\qquad \begin{cases} \psi \geq 0; & \\ \psi \text{ is convex}; & \\ \psi(tp) = |t|\,\psi(p) & \text{for all } t \in \mathrm{R},\, p \in \mathrm{R}^n; \\ a\,|p| \leq \psi(p) & \text{for some } a > 0 \text{ and for every } p \in \mathrm{R}^n. \end{cases}$$

If Ω is an open set in $\mathbf{R}^n$, we define, for $\varepsilon > 0$, the following set function:

$$\sigma_\varepsilon^\psi(\mathrm{E}\,;\,\Omega) = \inf\left\{ \int_\Omega \psi(\mathrm{D}u)\,\mathrm{d}x + \frac{1}{\varepsilon}\int_\Omega |u|\,\mathrm{d}x\,;\, u \in \mathrm{W}^{1,1}(\Omega),\right.$$
$$\left. u \text{ l.s.c. on } \overline{\Omega},\, u \geqq 1 \text{ on } \mathrm{E} \cap \overline{\Omega} \right\},$$

where E is an arbitrary subset of $\mathbf{R}^n$;
moreover let

$$\sigma^\Psi(\mathrm{E}\,;\,\Omega) = \lim_{\varepsilon \to 0+} \sigma_\varepsilon^\psi(\mathrm{E}\,;\,\Omega) = \sup_{\varepsilon > 0} \sigma_\varepsilon^\psi(\mathrm{E}\,;\,\Omega)\,.$$

Proposition 2. *$\sigma_\varepsilon^\psi(\cdot\,;\,\Omega)$ is an outer measure; $\sigma^\psi(\cdot\,;\,\Omega)$ is a measure on the Borel sets in $\mathbf{R}^n$.*

Example 1. If $\psi(p) = |p|$, $\Omega = \mathbf{R}^n$ and S is a portion of a smooth hypersurface, then

$$\sigma^\Psi(\mathrm{S}\,;\,\mathbf{R}^n) = 2\,\mathrm{H}^{n-1}(\mathrm{S})\,,$$

where H^{n-1} is the $(n-1)$-dimensional Hausdorff measure.

Example 2. If ψ verifies (3), if Ω is a bounded open set of $\mathbf{R}^n$ sufficiently smooth, if μ is given by $\mu(\mathrm{B}) = \mathrm{H}^{n-1}(\mathrm{B} \cap \partial\Omega)$ for every Borel set B in $\mathbf{R}^n$, and if, finally, $\nu(x)$ is the unit vector normal to $\partial\Omega$ at x, then we have

$$\frac{\mathrm{d}\sigma^\Psi}{\mathrm{d}\mu}(x) = \begin{cases} 0 & \text{if } x \notin \partial\Omega \\ \psi(\nu(x)) & \text{if } x \in \partial\Omega\,. \end{cases}$$

§ 3. Henceforth we shall denote by Ω a bounded open set of $\mathbf{R}^n$; by μ a Radon measure on the Borel sets of $\mathbf{R}^n$.

Let $f : \Omega \times \mathbf{R} \times \mathbf{R}^n \to [0, +\infty)$ be a Carathéodory function and $\psi : \mathbf{R}^n \to \mathbf{R}$ a function satisfying (3), such that

$$\psi(p) \leqq f(x, s, p) \quad \text{for all } x \in \Omega,\, p \in \mathbf{R}^n,\, s \in \mathbf{R}\,.$$

We shall assume

$$u \longrightarrow \int f(x, u, \mathrm{D}u)\, \mathrm{d}x$$

$\mathrm{L}^1(\mathrm{A})$ —l.s.c. on $\mathrm{W}^{1,1}(\Omega)$ for every open set $\mathrm{A} \subseteq \Omega$.

Semi-continuity Theorem.

Let Ω, f, ψ *and* μ *be as above.*

If $g : \overline{\Omega} \times \mathbf{R} \to [0, +\infty]$ *is a Borel function such that* $g(x, \cdot)$ *is l.s.c. for every* $x \in \overline{\Omega}$,

$$|g(x, u) - g(x, v)| \leq \frac{\mathrm{d}\sigma^{\psi}}{\mathrm{d}\mu}(x)|u - v|$$

for every $u, v \in \mathbf{R}$ *and* μ *—a.a.* $x \in \overline{\Omega}$ *such that* $\frac{\mathrm{d}\sigma^{\psi}}{\mathrm{d}\mu}(x) < +\infty$, *then the functional*

$$u \in \mathrm{W}^{1,1}(\Omega) \cap \mathrm{C}^0(\overline{\Omega}) \longrightarrow \mathrm{F}(u) = \int_{\Omega} f(x, u, \mathrm{D}u)\, \mathrm{d}x + \int_{\overline{\Omega}} g(x, u)\, d\mu$$

is $\mathrm{L}^1(\Omega)$ *—l.s.c.* .

Taking into account the function $\frac{\mathrm{d}\sigma^{\psi}}{\mathrm{d}\mu}$, we give also a first answer to the problem (P 2).

If $f = f(x, p)$ is a Carathéodory function such that

$$\psi(p) \leqq f(x, p) \qquad \text{for all } x \in \Omega, p \in \mathbf{R}^n,$$

$$f(x, p + q) \leqq f(x, p) + \psi(q) \qquad \text{for all } x \in \Omega, p, q \in \mathbf{R}^n,$$

$u \longrightarrow \int_{\mathrm{A}} f(x, \mathrm{D}u)\, \mathrm{d}x$ is $\mathrm{L}^1(\mathrm{A})$ —l.s.c. on $\mathrm{W}^{1,1}(\Omega)$ for every open set $\mathrm{A} \subseteq \Omega$, and if, moreover, Ω is sufficiently smooth, then the following theorem holds:

Relaxation Theorem.

Let f, Ψ *and* Ω *be as above.*

If $g : \overline{\Omega} \times \mathbf{R} \to [0, +\infty)$ *is a Carathéodory function such that* $g(x, s) \leqq \leqq a(x)\, \Phi(|s|)$ *with* $a \in \mathrm{L}^1(d\mu)$, Φ *increasing, then we have:*

$$sc^-(\mathrm{L}^1(\Omega))\left[\int_{\Omega} f(x, \mathrm{D}u)\, \mathrm{d}x + \int_{\overline{\Omega}} g(x, u)\, \mathrm{d}\mu\right] = \int_{\Omega} f(x, \mathrm{D}u)\, \mathrm{d}x + \int_{\overline{\Omega}} \gamma(x, u)\, \mathrm{d}\mu,$$

where

$$\gamma(x\,,s) = \begin{cases} \inf\limits_{t\in\mathbf{R}} \left\{ g(x\,,t) + \dfrac{d\sigma^{\psi}}{d\mu}(x)\,|\,s-t\,| \right\} & \text{if } \dfrac{d\sigma^{\psi}}{d\mu}(x) < +\infty\,, \\ g(x\,,s) & \text{if } \dfrac{d\sigma^{\psi}}{d\mu}(x) = +\infty\,. \end{cases}$$

Example 3. The relaxation formula just given can, for instance, be used for the functional

$$A(u) = \int_{\Omega} \sqrt{1 + |\,Du\,|^2}\,dx + \int_{\overline{\Omega}} g(x\,,u)\,d\mu$$

that one meets when studying Plateau's problem.

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