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Analyticity of the Spectral Multi-Function in Topological Algebras

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Geometria. — *Analyticity of the Spectral Multi-Function in Topological Algebras.* Nota di ENRICO CASADIO TARABUSI (*), presentata (**) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Se $f : \Omega \to \mathscr{U}$ è un'applicazione olomorfa di un dominio di $\mathbf{C}$ in un'algebra topologica che gode di certe proprietà, si dimostra che la multifunzione «spettro» $\sigma \circ f : \Omega \to 2^{\mathbf{C}}$ è analitica secondo Oka.

1. INTRODUCTION

Let Ω be a domain in $\mathbf{C}$, $\mathbf{B}$ a complex Banach space, $\mathscr{L}(\mathbf{B})$ the complex Banach algebra of continuous endomorphisms of $\mathbf{B}$, $2^{\mathbf{C}}$ the family of subsets of $\mathbf{C}$, $\sigma : \mathscr{L}(\mathbf{B}) \to 2^{\mathbf{C}}$ the multifunction "spectrum" (mapping $x \in \mathscr{L}(\mathbf{B})$ into its spectrum $\sigma(x)$). According to [7, p. 371, Corollary 3.3.], if $f : \Omega \to \mathscr{L}(\mathbf{B})$ is a holomorphic map, then the multifunction $\sigma \circ f : \Omega \to 2^{\mathbf{C}}$ is analytic in the sense of Oka (i.e. $\sigma \circ f$ is upper semi-continuous and each connected component of the open set $D = \{(\lambda, z) \in \Omega \times \mathbf{C} \mid z \notin \sigma(f(\lambda))\}$ in $\mathbf{C}^2$ is a domain of holomorphy). A partial converse to the above statement is also proved in the same paper [p. 365, Theorem IV.]: given an analytic multifunction $\sigma : \Omega \to 2^{\mathbf{C}}$ (taking its values among the non–empty compact subsets of $\mathbf{C}$), there exist a complex Hilbert space $\mathscr{H}$ and a holomorphic map $f : \Omega \to \mathscr{L}(\mathscr{H})$ such that $s \equiv \sigma \circ f$ on Ω, *provided* Ω is bounded and s is uniformly bounded on Ω (i.e. $\sup_{\lambda \in \Omega} \max_{z \in s(\lambda)} |z| < \infty$). We shall show here that the Oka-analyticity of the spectrum is a property of a class of topological algebras which is larger than that of Banach algebras.

2. SEMI-CONTINUITY OF THE SPECTRUM

We shall give first a characterization of complex topological algebras whose spectrum is upper semi-continuous, thus establishing the reverse implication of [11, p. 63, Lemma 5.2.]. Let us recall some definitions.

DEFINITION 1. *Let $\mathscr{U}$ be a complex topological algebra. A multifunction $\Sigma : \mathscr{U} \to 2^{\mathbf{C}}$ is said to be* upper semi-continuos (u.s.c.) *if, for any $x \in \mathscr{U}$ and any neighbourhood* A *of $\Sigma(x)$ in $\mathbf{C}$ there exists $U \in N_0$ (N_0 being the family of 0-neighbourhoods in $\mathscr{U}$) such that $y \in \mathscr{U}$ implies $\Sigma(x + y) \subseteq A$.*

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Let $\mathcal{U}$ be a complex algebra. Set $x \circ y = x + y - x.y$, for any $x, y \in \mathcal{U}$: an element $x \in \mathcal{U}$ is *quasi-regular* (*q. r.*) if a (unique) $x' \in \mathcal{U}$ exists so that $x \circ x' = x' \circ x = 0$ (x' is the *quasi-inverse* of x); and $\mathcal{U}'$ is the set of q.r. elements of $\mathcal{U}$. If $\mathcal{U}$ has an identity element e (i.e. $\mathcal{U}$ is *unital*), then $e - x \circ y = (e - x).(e - y)$ for every $x, y \in \mathcal{U}$: thus x is q.r. if and only if $e - x$ is invertible, in which case $(e - x)^{-1} = e - x'$. So, if $\mathcal{U}^{-1}$ is the set of invertible elements in $\mathcal{U}$, then $\mathcal{U}^{-1} = e - \mathcal{U}'$.

The spectrum $\sigma_{\mathcal{U}}(x) = \sigma(x)$ in $\mathcal{U}$ of an element $x \in \mathcal{U}$ is thus defined: $\sigma(x) \setminus \{0\} = \{z \in \mathbf{C}^* \mid \frac{x}{z} \notin \mathcal{U}'\}$; and $0 \in \sigma(x)$ if and only if $\mathcal{U}$ is unital and $x \in \mathcal{U}^{-1}$. If $\mathcal{U}$ is unital we have $\sigma(x) = \{z \in \mathbf{C} \mid ze - x \in \mathcal{U}^{-1}\}$.

DEFINITION 2. *A complex topological algebra $\mathcal{U}$ is said to be* Q *if $\mathcal{U}'$ (or, equivalently, $\mathcal{U}^{-1}$ if $\mathcal{U}$ is unital) is open in $\mathcal{U}$.*

If $\mathcal{U}$ is a complex algebra without identity element, let $\check{\mathcal{U}}$ be the complex algebra $\mathcal{U} \oplus \mathbf{C}$ where $(x, \mu).(y, \nu) = (x.y + \nu x + \mu y, \mu\nu)$, for any (x, μ), $(y, \nu) \in \check{\mathcal{U}}$. Thus $\check{\mathcal{U}}$ is unital, (0,1) being its identity element; and $\mathcal{U}$ is identified with its two-sided regular maximal ideal $\mathcal{U} \times \{0\}$. Moreover $\check{\mathcal{U}}' = \left\{(x, \mu) \in \check{\mathcal{U}} \mid \mu \neq 1, \frac{x}{1-\mu} \in \mathcal{U}'\right\}$ (in particular, $\check{\mathcal{U}}' \cap \mathcal{U} = \mathcal{U}'$): if $(x, \mu) \in \check{\mathcal{U}}'$, then $(x, \mu)' = \left(\frac{1}{1-\mu}\left(\frac{x}{1-\mu}\right)', \frac{\mu}{\mu - 1}\right)$ (in particular, $(x, 0)' = (x', 0)$ for every $x \in \mathcal{U}'$). So $\sigma_{\check{\mathcal{U}}}\big((x, \mu)\big) = \sigma_{\mathcal{U}}(x) + \mu$ for every $(x, \mu) \in \check{\mathcal{U}}$ (in particular, $\sigma_{\check{\mathcal{U}}}\big((x, 0)\big) = \sigma_{\mathcal{U}}(x)$ for every $x \in \mathcal{U}$). It is thus evident that, if $\mathcal{U}$ is also a topological algebra ($\check{\mathcal{U}}$ has then the product topology, and $\mathcal{U}$ is closed in $\check{\mathcal{U}}$), $\mathcal{U}$ is Q if and only if $\check{\mathcal{U}}$ is; and the multifunction spectrum $\sigma_{\mathcal{U}} : \mathcal{U} \to 2^{\mathbf{C}}$ is u.s.c. if and only if $\sigma_{\check{\mathcal{U}}} : \check{\mathcal{U}} \to 2^{\mathbf{C}}$ is. Moreover, the map $x \mapsto x' : \mathcal{U}' \to \mathcal{U}$ is continuous if and only if $(x, \mu) \mapsto (x, \mu)' : \check{\mathcal{U}}' \to \check{\mathcal{U}}$ is: this fact will be used in Theorem 6. below. Therefore we shall only consider *unital* algebras in our proofs.

PROPOSITION 3. *Let $\mathcal{U}$ be a complex topological algebra. Then $\mathcal{U}$* is Q *if and only if the multifunction spectrum is u.s.c. on $\mathcal{U}$.*

Proof. If $\mathcal{U}$ is not Q, then (see [4, p. 77, Lemma E.2.]) $\mathcal{U}'$ has empty interior. Therefore, for any $U \in N_0$ there exists $x \in U \setminus \mathcal{U}'$: that is to say, $1 \in \sigma(x)$. But $\sigma(0) = \{0\}$.

Conversely, suppose $\mathcal{U}$ is (unital and) Q, and let $x \in \mathcal{U}$: then (see [4, p. 77, Lemma E.3.]) $\sigma(x)$ is a compact subset of $\mathbf{C}$. It will suffice to show that for any $\varepsilon > 0$ there exists $U \in N_0$ such that $y \in U$ implies $\sigma(x + y) \subseteq A_\varepsilon$, where A_ε is the open set $\{z \in \mathbf{C} \mid \text{dist}(z, \sigma(x)) < \varepsilon\}$ (A_ε is empty if so is $\sigma(x)$).

Let us first prove the existence of $R > 0$ and $U_\infty \in N_0$ such that $y \in U_\infty$ implies $\sigma(x+y) \subseteq \overline{B(0,R)}$ (we shall denote with $B(z,r)$ the open ball $\{\zeta \in \mathbf{C} \mid |\zeta - z| < r\}$). Since $0 \in \mathcal{U}'$, there exists $V_\infty \in N_0$ such that $V_\infty \subseteq \mathcal{U}'$. Therefore there exist: a balanced $U_\infty \in N_0$ such that $U_\infty + U_\infty \subseteq V_\infty$; and a positive $r_\infty < 1$ such that $w \in B(0, r_\infty)$ implies $wx \in U_\infty$. Set $R = \frac{1}{r_\infty}$: if $y \in U$ and $z \in \mathbf{C} \setminus \overline{B(0,R)}$, we have $\left|\frac{1}{z}\right| < \frac{1}{R} = r_\infty \leq 1$, so $\frac{x+y}{z} = \frac{1}{z}x + \frac{1}{z}y \in U_\infty + \frac{1}{z}U_\infty \subseteq U_\infty + U_\infty \subseteq V_\infty \subseteq \mathcal{U}'$, that is, $z \notin \sigma(x+y)$.

Now let $\varepsilon > 0$, and set $K = \overline{B(0,R)} \setminus A_\varepsilon$. If $z \in K$, then $x - ze \in \mathcal{U}^{-1}$; therefore $V_z \in N_0$ exists so that $x - ze + V_z \subseteq \mathcal{U}^{-1}$. As above, let $U_z \in N_0$ be such that $U_z + U_z \subseteq V_z$; and let $r_z > 0$ be such that $w \in B(0, r_z)$ implies $we \in U_z$. Thus, if $y \in U_z$ and $\zeta \in B(z, r_z)$, then $(x+y) - \zeta e = (x - ze) + y + (z - \zeta)e \in (x - ze) + U_z + U_z \subseteq x - ze + V_z \subseteq \mathcal{U}^{-1}$, that is, $\zeta \notin \sigma(x+y)$.

But K is compact, so from its open covering $\{B(z, r_z)\}_{z \in K}$ a finite subcovering $\{B(z_j, r_{z_j})\}_{j=1,\dots,N}$ (where $z_1, \dots, z_N \in K$) can be extracted. Set $U = U_\infty \cap \left(\bigcap_{j=1}^{N} U_{z_j}\right)$: then $U \in N_0$, and $\sigma(x+y) \subseteq A_\varepsilon$ whenever $y \in U$. ∎

REMARK 4. The "if" part of Proposition 3. can be so sharpened: if $\mathcal{U}$ is not Q, then, for any $x \in \mathcal{U}$, $z \in \mathbf{C}$, and $U \in N_0$, there exists $y \in U$ such that $z \in \sigma(x+y)$. In fact (we assume $z \neq 0$: the case $z = 0$ being straightforward) $V = \frac{1}{z}U$ is still in N_0: if $y_1 \in V$ is such that $\frac{x}{z} + y_1 \notin \mathcal{U}'$, let $y = zy_1 \in zV = U$. Thus $\frac{x+y}{z} = \frac{x}{z} + y_1 \notin \mathcal{U}'$, that is, $z \in \sigma(x+y)$. ∎

3. THE MAIN RESULT

Let us start with a definition.

DEFINITION 5. *A (complex) topological algebra* $\mathcal{U}$ *is said to* have continuous quasi-inversion *if it is* Q, *and the map* $x \mapsto x' : \mathcal{U}' \to \mathcal{U}$ (*or, equivalently if* $\mathcal{U}$ *is unital,* $x \mapsto x^{-1} : \mathcal{U}^{-1} \to \mathcal{U}$) *is continuous.*

For example, a locally multiplicatively-convex Q-algebra has continuous quasi-inversion: cfr. [4, p. 10, Proposition 2.8.].

THEOREM 6. *Let* $\mathcal{U}$ *be a complex locally convex algebra having continuous quasi-inversion. Then, for any domain* Ω *in* $\mathbf{C}$ *and any holomorphic map* $f : \Omega \to \mathcal{U}$, *the multifunction* $\sigma \circ f : \Omega \to 2^{\mathbf{C}}$ *is Oka-analytic.*

REMARK 7. *a*) As is customary, no assumption is made on the continuity of the product in a topological algebra, or on the completeness of the algebra itself.

b) A map $f : \Omega \to \mathscr{U}$ is said to be *holomorphic* when, for any $z \in \Omega$, the limit $f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$ exists in $\tilde{\mathscr{U}}$ (the completion of $\mathscr{U}$ as a topological vector space; see [2, p. 59, Definition 2.]. For other definitions, see [5, p. 22, Théorème 1.2.2.]: however, in the present case they are all equivalent to ours).

Proof of Theorem 6. By Proposition 3. and the continuity of f, the multifunction $\sigma \circ f$ is u.s.c. on Ω (besides, D is then open in $\mathbf{C}^2$). We shall assume $\mathscr{U}$ to be unital (cfr. § 2.). Let $(\lambda_0, z_0) \in (\Omega \times \mathbf{C}) \setminus D$ (so $z_0 \in \sigma(f(\lambda_0))$) and $B = \mathbf{C}(x_0)$ be the subalgebra of $\mathscr{U}$ of quotients of polynomials in $x_0 = f(\lambda_0)$, with complex coefficients, by invertible polynomials of the same kind. Obviously if such a quotient has an inverse, that is still a quotient of the same kind, that is, $\mathscr{U}^{-1} \cap B = B^{-1}$. Thus the complex unital locally convex algebra B is also Q, and $x \mapsto x^{-1} : B^{-1} \to B$ is continuous, i.e. B has continuous quasi-inversion. Moreover B is commutative. By Zorn's lemma, there exists a maximal ideal $\mathfrak{m}$ in B containing $y_0 = ze - x_0$ (of course $\mathfrak{m}$ is regular). Since B is Q, $\mathfrak{m}$ is also closed in B (cfr. [4, p. 77, Lemma E. 4.]); and, since B has the other properties listed above, the Gel'fand-Mazur theorem (see [3, p. 811]) can be applied to infer that the topological algebra $B/\mathfrak{m}$ is isomorphic to $\mathbf{C}$. In other words we have a non-zero continuous linear multiplicative functional $\varphi : B \to \mathbf{C}$ such that $\varphi(y_0) = 0$. So for every $z \in \mathbf{C} \setminus \sigma(x_0)$ we have

$$\varphi([ze - x_0]^{-1}) = \frac{1}{\varphi(ze - x_0)} = \frac{1}{(z - z_0)\,\varphi(e) - \varphi(y_0)} = \frac{1}{z - z_0}.$$

We can now apply to φ in $\mathscr{U}$ the Hahn-Banach theorem, $\mathscr{U}$ being locally convex. Thus, let $\tilde{\varphi} : \mathscr{U} \to \mathbf{C}$ be a continuous linear (but not necessarily multiplicative) functional that extends φ; and set $\psi : D \to \mathscr{U}$ by $\psi(\lambda, z) = [ze - f(\lambda)]^{-1}$ for every $(\lambda, z) \in D$. If we prove that $b = \tilde{\varphi} \circ \psi : D \to \mathbf{C}$ is holomorphic, then we shall apply the criterion given by [8, p. 14, Lemma 2.] to conclude that each connected component of D is a domain of holomorphy.

Let $J : \mathscr{U} \times \mathscr{U} \to \mathscr{U}$ be the Jordan product $J(x, y) = \frac{x.y + y.x}{2}$, for any $x, y \in \mathscr{U}$: $\mathscr{U}$ having continuous quasi-inversion, by [10, p. 1686, Proposition 1.] J is jointly continuous. Thus if $(\lambda, z) \in D$ the following limit exists:

$$\frac{\partial \psi}{\partial z}(\lambda, z) = \lim_{h \to 0} \frac{\psi(\lambda, z + h) - \psi(\lambda, z)}{h} =$$

$$= \lim_{h \to 0} \frac{[(z + h)\,e - f(\lambda)]^{-1} - [ze - f(\lambda)]^{-1}}{h} =$$

$$= -\lim_{h \to 0} J\left([(z + h)\,e - f(\lambda)]^{-1}, [ze - f(\lambda)]^{-1}\right) =$$

$$= -J\left([ze - f(\lambda)]^{-1}, [ze - f(\lambda)]^{-1}\right) = -[ze - f(\lambda)]^{-2}$$

(we have used the equality $x^{-1} - y^{-1} = \frac{x^{-1}.(y-x).y^{-1} + y^{-1}.(y-x).x^{-1}}{2}$, true for any $x, y \in \mathscr{U}^{-1}$).

To prove the holomorphicity of ψ in the variable λ, we need to extend J to $\tilde{J} : \tilde{\mathscr{U}} \times \mathscr{U} \to \tilde{\mathscr{U}}$ in a *jointly* continuous fashion. Indeed, for every $y \in \mathscr{U}$ the map $J_y : \mathscr{U} \to \mathscr{U}$ given by $J_y(x) = J(x, y)$ is continuous and linear, therefore it extends to a continuous linear map $\tilde{J}_y : \tilde{\mathscr{U}} \to \tilde{\mathscr{U}}$: set $\tilde{J}(x, y) = \tilde{J}_y(x)$ for any $(x, y) \in \tilde{\mathscr{U}} \times \mathscr{U}$. Let now $\tilde{U}$ be a closed 0-neighbourhood in $\tilde{\mathscr{U}}$ (a topological vector space always admits a fundamental system of 0-neighbourhoods, cfr. [6, p. 16, 1.3.]): then $U = \tilde{U} \cap \mathscr{U}$ is a 0-neighbourhood in $\mathscr{U}$, thus a 0-neighbourhood V in $\mathscr{U}$ exists so that $J(V \times V) \subseteq U$. If $\bar{V}$ is the closure of V in $\tilde{\mathscr{U}}$, then $\bar{V}$ is a 0-neighbourhood in $\tilde{\mathscr{U}}$ (cfr. [6, p. 17, 1.5.]), and $\tilde{J}(\bar{V} \times V) \subseteq \tilde{U}$: in fact, if $y \in V$, then $J_y(V) \subseteq U$, whence $\tilde{J}_y(\bar{V}) \subseteq \tilde{U}$.

Now let $(\lambda, z) \in D$, and set $g : B(0, \delta) \to \mathscr{U}^{-1}$ by $g(h) = ze - f(\lambda + h)$ ($\delta > 0$ small enough): g is holomorphic in 0. If $a(h) = \frac{g(h) - g(0)}{h}$ for every $h \in B(0, \delta)$, then an easy computation leads to:

$$\frac{g(h)^{-1} - g(0)^{-1}}{h} = -\frac{g(h)^{-1}.a(h).g(0)^{-1} + g(0)^{-1}.a(h).g(h)^{-1}}{h} =$$

$$= J\left(a(h), J\left(g(h)^{-1}, g(0)^{-1}\right)\right) - J\left(J\left(a(h), g(h)^{-1}\right), g(0)^{-1}\right) -$$

$$- J\left(J\left(a(h), g(0)^{-1}\right), g(h)^{-1}\right).$$

Since $\lim_{h \to 0} g(h)^{-1} = g(0)^{-1} \in \mathscr{U}$, while $\lim_{h \to 0} a(h) = g'(0) \in \tilde{\mathscr{U}}$, the following limit exists in $\tilde{\mathscr{U}}$:

$$\frac{\partial \psi}{\partial \lambda}(\lambda, z) = \lim_{h \to 0} \frac{g(h)^{-1} - g(0)^{-1}}{h} = \tilde{J}\left(g'(0), g(0)^{-2}\right) -$$

$$- 2\,\tilde{J}\left(\tilde{J}\left(g'(0), g(0)^{-1}\right), g(0)^{-1}\right)$$

(if we could expand $\tilde{J}$, the latter expression would of course equal $-g(0)^{-1} . .g'(0).g(0)^{-1}$).

Therefore, b is separately holomorphic in D: (it being continuous on D) b is then (jointly) holomorphic in D. ∎

Theorem 6. has several consequences. Among them are the logarithmic pluri-sub-harmonicity of several functions of the spectrum in $\mathscr{U}$, such as the spectral radius, any k-th spectral diameter (with $k \in \mathbf{N}$), the spectral capacity, and many others. Also, we have: the pluri-analyticity of isolated eigenvalues, and, more generally, of spectral sets; the finite scarcity and countable scarcity theorems; and so on. For deeper analyses of the consequences of the Oka-analyticity, see e.g. [1], [9], [11], and the literature cited there.

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