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Analyticity of the Spectral Multi-Function in Topological Algebras

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RIASSUNTO. — Se \( f : \Omega \rightarrow \mathcal{H} \) è un’applicazione olomorfa di un dominio di \( \mathbf{C} \) in un’algebra topologica che gode di certe proprietà, si dimostra che la multifunzione “spettro” \( \sigma_f : \Omega \rightarrow 2^\mathbf{C} \) è analitica secondo Oka.

1. INTRODUCTION

Let \( \Omega \) be a domain in \( \mathbf{C} \), \( \mathbf{B} \) a complex Banach space, \( \mathcal{L}(\mathbf{B}) \) the complex Banach algebra of continuous endomorphisms of \( \mathbf{B} \), \( 2^\mathbf{C} \) the family of subsets of \( \mathbf{C} \), \( \sigma : \mathcal{L}(\mathbf{B}) \rightarrow 2^\mathbf{C} \) the multifunction “spectrum” (mapping \( x \in \mathcal{L}(\mathbf{B}) \) into its spectrum \( \sigma(x) \)). According to [7, p. 371, Corollary 3.3.], if \( f : \Omega \rightarrow \mathcal{L}(\mathbf{B}) \) is a holomorphic map, then the multifunction \( \sigma \circ f : \Omega \rightarrow 2^\mathbf{C} \) is analytic in the sense of Oka (i.e. \( \sigma \circ f \) is upper semi-continuous and each connected component of the open set \( D = \{(\lambda, z) \in \Omega \times \mathbf{C} | z \in \sigma(f(\lambda))\} \) in \( \mathbf{C}^2 \) is a domain of holomorphy). A partial converse to the above statement is also proved in the same paper [p. 365, Theorem IV.]: given an analytic multifunction \( \sigma : \Omega \rightarrow 2^\mathbf{C} \) (taking its values among the non-empty compact subsets of \( \mathbf{C} \)), there exist a complex Hilbert space \( \mathcal{H} \) and a holomorphic map \( f : \Omega \rightarrow \mathcal{L}(\mathcal{H}) \) such that \( s = \sigma \circ f \) on \( \Omega \), provided \( \Omega \) is bounded and \( s \) is uniformly bounded on \( \Omega \) (i.e. \( \sup_{\lambda \in \Omega} \max_{z \in \sigma(f(\lambda))} |z| < \infty \)). We shall show here that the Oka-analyticity of the spectrum is a property of a class of topological algebras which is larger than that of Banach algebras.

2. SEMI-CONTINUITY OF THE SPECTRUM

We shall give first a characterization of complex topological algebras whose spectrum is upper semi-continuous, thus establishing the reverse implication of [11, p. 63, Lemma 5.2.]. Let us recall some definitions.

DEFINITION 1. Let \( \mathcal{A} \) be a complex topological algebra. A multifunction \( \Sigma : \mathcal{A} \rightarrow 2^\mathbf{C} \) is said to be upper semi-continuous (u.s.c.) if, for any \( x \in \mathcal{A} \) and any neighbourhood \( A \) of \( \Sigma(x) \) in \( \mathbf{C} \), there exists \( U \in \mathcal{N}_0 \) (\( \mathcal{N}_0 \) being the family of \( 0 \)-neighbourhoods in \( \mathcal{A} \)) such that \( y \in \mathcal{A} \) implies \( \Sigma(x + y) \subseteq A \).

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Let \( \mathcal{U} \) be a complex algebra. Set \( x \circ y = x + y - x \cdot y \), for any \( x, y \in \mathcal{U} \); an element \( x \in \mathcal{U} \) is quasi-regular (q.r.) if a (unique) \( x' \in \mathcal{U} \) exists so that \( x \circ x' = x' \circ x = 0 \) (\( x' \) is the quasi-inverse of \( x \)); and \( \mathcal{U}' \) is the set of q.r. elements of \( \mathcal{U} \). If \( \mathcal{U} \) has an identity element \( e \) (i.e. \( \mathcal{U} \) is unital), then \( e - x \circ y = (e - y) \) for every \( x, y \in \mathcal{U} \): thus \( x \) is q.r. if and only if \( e - x \) is invertible, in which case \( (e - x)^{-1} = e - x' \). So, if \( \mathcal{U}^{-1} \) is the set of invertible elements in \( \mathcal{U} \), then \( \mathcal{U}^{-1} = e - \mathcal{U}' \).

The spectrum \( \sigma_{\mathcal{U}}(x) = \sigma(x) \) in \( \mathcal{U} \) of an element \( x \in \mathcal{U} \) is thus defined: \( \sigma(x) \setminus \{0\} = \{z \in \mathbb{C}^* | \frac{x}{z} \in \mathcal{U}'\} \); and \( 0 \in \sigma(x) \) if and only if \( \mathcal{U} \) is unital and \( x \in \mathcal{U}^{-1} \). If \( \mathcal{U} \) is unital we have \( \sigma(x) = \{z \in \mathbb{C} | ze - x \in \mathcal{U}^{-1}\} \).

**Definition 2.** A complex topological algebra \( \mathcal{U} \) is said to be Q if \( \mathcal{U}' \) (or, equivalently, \( \mathcal{U}^{-1} \) if \( \mathcal{U} \) is unital) is open in \( \mathcal{U} \).

If \( \mathcal{U} \) is a complex algebra without identity element, let \( \hat{\mathcal{U}} \) be the complex algebra \( \mathcal{U} \oplus \mathbb{C} \) where \( (x, \mu), (y, \nu) = (x \cdot y + \mu y, \mu y) \), for any \( (x, \mu), (y, \nu) \in \hat{\mathcal{U}} \). Thus \( \mathcal{U} \) is unital, \((0,1)\) being its identity element; and \( \hat{\mathcal{U}} \) is identified with its two-sided regular maximal ideal \( \mathcal{U} \times \{0\} \). Moreover \( \hat{\mathcal{U}}' = \left\{ (x, \mu) \in \hat{\mathcal{U}} | \mu \neq 1, \frac{x}{1 - \mu} \in \mathcal{U}' \right\} \) (in particular, \( \hat{\mathcal{U}}' \cap \mathcal{U} = \mathcal{U}' \)): if \( (x, \mu) \in \mathcal{U}' \), then \( (x, \mu)' = \left( \frac{1}{1 - \mu} \left( x \right)' - \frac{\mu}{\mu - 1} \right) \) (in particular, \( (x, 0)' = (x', 0) \) for every \( x \in \mathcal{U}' \)). So \( \sigma_{\hat{\mathcal{U}}}(x, \mu) = \sigma_{\mathcal{U}}(x) + \mu \) for every \( (x, \mu) \in \hat{\mathcal{U}} \) (in particular, \( \sigma_{\hat{\mathcal{U}}}(x, 0) = \sigma_{\mathcal{U}}(x) \) for every \( x \in \mathcal{U} \)). It is thus evident that, if \( \mathcal{U} \) is also a topological algebra (\( \hat{\mathcal{U}} \) has then the product topology, and \( \mathcal{U} \) is closed in \( \hat{\mathcal{U}} \), \( \mathcal{U} \) is Q if and only if \( \hat{\mathcal{U}} \) is; and the multifunction spectrum \( \sigma_{\hat{\mathcal{U}}} : \mathcal{U} \to 2^\mathbb{C} \) is u.s.c. if and only if \( \sigma_{\hat{\mathcal{U}}} : \mathcal{U} \to 2^\mathbb{C} \) is. Moreover, the map \( x \mapsto x' : \mathcal{U}' \to \mathcal{U} \) is continuous if and only if \( (x, \mu) \mapsto (x, \mu)' : \mathcal{U}' \to \mathcal{U} \) is; this fact will be used in Theorem 6. below. Therefore we shall only consider unital algebras in our proofs.

**Proposition 3.** Let \( \mathcal{U} \) be a complex topological algebra. Then \( \mathcal{U} \) is Q if and only if the multifunction spectrum is u.s.c. on \( \mathcal{U} \).

**Proof.** If \( \mathcal{U} \) is not Q, then (see [4, p. 77, Lemma E.2.]) \( \mathcal{U}' \) has empty interior. Therefore, for any \( U \in \mathcal{N}_0 \) there exists \( x \in U \setminus \mathcal{U}' \), that is to say, \( 1 \in \sigma(x) \). But \( \sigma(0) = \{0\} \).

Conversely, suppose \( \mathcal{U} \) is (unital and) Q, and let \( x \in \mathcal{U} \); then (see [4, p. 77, Lemma E.3.]) \( \sigma(x) \) is a compact subset of \( \mathbb{C} \). It will suffice to show that for any \( \varepsilon > 0 \) there exists \( U \in \mathcal{N}_0 \) such that \( y \in U \) implies \( \sigma(x + y) \subseteq A_\varepsilon \), where \( A_\varepsilon \) is the open set \( \{z \in \mathbb{C} | \text{dist}(z, \sigma(x)) < \varepsilon\} \). (\( A_\varepsilon \) is empty if so is \( \sigma(x) \)).
Let us first prove the existence of $R > 0$ and $U \subseteq N_0$ such that $y \in U \implies G(x + y) \subseteq B(0, R)$ (we shall denote with $B(0, R)$ the open ball for $z \in C \mid |z - z| < r$). Since $0 \in \mathcal{U}'$, there exists $V_\infty \subseteq N_0$ such that $V_\infty \subseteq \mathcal{U}'$. Therefore there exist: a balanced $U_\infty \subseteq N_0$ such that $U_\infty + U_\infty \subseteq V_\infty$, and a positive $r_\infty < 1$ such that $w \in B(0, r_\infty)$ implies $wx \in U_\infty$. Set $R = \frac{1}{r_\infty}$: if $y \in U$ and $x \in C \setminus B(0, R)$, we have $\left| \frac{1}{z} \right| < \frac{1}{R} = r_\infty \leq 1$, so $\frac{x + y}{z} = \frac{1}{z}x + \frac{1}{z}y \in U_\infty + \frac{1}{z}U_\infty \subseteq U_\infty + U_\infty \subseteq V_\infty \subseteq \mathcal{U}'$, that is, $z \in \sigma(x + y)$.

Now let $\varepsilon > 0$, and set $K = \overline{B(0, R) \setminus A_x}$. If $z \in K$, then $x - z \in \mathcal{U}^{-1}$, therefore $V_z \subseteq N_0$ exists so that $x - ze + V_z \subseteq \mathcal{U}^{-1}$. As above, let $U_z \subseteq N_0$ be such that $U_z + U_z \subseteq V_z$; and let $r_z > 0$ be such that $w \in B(0, r_z)$ implies $we \in U_z$. Thus, if $y \in U_z$ and $x \in B(z, r_z)$, then $(x + y) - ze = (x - ze) + y + (z - z)e \in (x - ze) + U_z + U_z \subseteq x - ze + V_z \subseteq \mathcal{U}^{-1}$, that is, $z \in \sigma(x + y)$.

But $K$ is compact, so from its open covering $\{B(z_j, r_{z_j})\}_{j=1,\ldots,N}$ a finite sub-covering $\{U_j\}_{j=1,\ldots,N}$ (where $z_1, \ldots, z_N \in K$) can be extracted. Set $U = U_\infty \cap \left( \bigcap_{j=1}^{N} U_j \right)$: then $U \subseteq N_0$, and $\sigma(x + y) \subseteq A_z$ whenever $y \in U$.

Remark 4. The "if" part of Proposition 3. can be so sharpened: if $\mathcal{U}$ is not $Q$, then for any $x \in \mathcal{U}$, $z \in C$, and $U \subseteq N_0$, there exists $y \in U$ such that $z \in \sigma(x + y)$. In fact (we assume $z \neq 0$: the case $z = 0$ being straightforward) $V = \frac{1}{z}U$ is still in $N_0$: if $y_1 \in V$ is such that $\frac{x}{z} + y_1 \in \mathcal{U}'$, let $y = zy_1 \in \mathcal{U}' = U$. Thus $\frac{x + y}{z} = \frac{x}{z} + y_1 \in \mathcal{U}'$, that is, $z \in \sigma(x + y)$.

3. THE MAIN RESULT

Let us start with a definition.

Definition 5. A (complex) topological algebra $\mathcal{U}$ is said to have continuous quasi-inversion if it is $Q$, and the map $x \mapsto x' : \mathcal{U} \rightarrow \mathcal{U}$ (or, equivalently if $\mathcal{U}$ is unital, $x \mapsto x^{-1} : \mathcal{U}^{-1} \rightarrow \mathcal{U}$) is continuous.

For example, a locally multiplicatively-convex $Q$-algebra has continuous quasi-inversion: cf. [4, p. 10, Proposition 2.8].

Theorem 6. Let $\mathcal{U}$ be a complex locally convex algebra having continuous quasi-inversion. Then, for any domain $\Omega$ in $C$ and any holomorphic map $f : \Omega \rightarrow \mathcal{U}$, the multifunction $\sigma \circ f : \Omega \rightarrow 2^C$ is Oka-analytic.

Remark 7. a) As is customary, no assumption is made on the continuity of the product in a topological algebra, or on the completeness of the algebra itself.
b) A map $f : \Omega \to \mathcal{U}$ is said to be holomorphic when, for any $z \in \Omega$, the limit $f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$ exists in $\mathcal{U}$ (the completion of $\mathcal{U}$ as a topological vector space; see [2, p. 59, Definition 2.]). For other definitions, see [5, p. 22, Théorème 1.2.2.]; however, in the present case they are all equivalent to ours.

**Proof of Theorem 6.** By Proposition 3. and the continuity of $f$, the multifunction $\sigma \circ f$ is u.s.c. on $\Omega$ (besides, $D$ is then open in $\mathbb{C}^n$). We shall assume $\mathcal{U}$ to be unital (cfr. § 2.). Let $(\lambda_0, z_0) \in (\Omega \times \mathbb{C}) \setminus D$ (so $z_0 \in \sigma (f (\lambda_0))$) and $B = \mathcal{U} (x_0)$ be the subalgebra of $\mathcal{U}$ of quotients of polynomials in $x_0 = f (\lambda_0)$, with complex coefficients, by invertible polynomials of the same kind. Obviously if such a quotient has an inverse, that is still a quotient of the same kind, that is, $\mathcal{U}^{-1} \cap B = B^{-1}$. Thus the complex unital locally convex algebra $B$ is also $\mathcal{Q}$, and $x \mapsto x^{-1} : B^{-1} \to B$ is continuous, i.e. $B$ has continuous quasi-inversion. Moreover $B$ is commutative. By Zorn's lemma, there exists a maximal ideal $m$ in $B$ containing $y_0 = ze - x_0$ (of course $m$ is regular). Since $B$ is $\mathcal{Q}$, $m$ is also closed in $B$ (cfr. [4, p. 77, Lemma E. 4.]); and, since $B$ has the other properties listed above, the Gel'fand-Mazur theorem (see [3, p. 81]) can be applied to infer that the topological algebra $B/m$ is isomorphic to $\mathbb{C}$. In other words we have a non-zero continuous linear multiplicative functional $\varphi : B \to \mathbb{C}$ such that $\varphi (y_0) = 0$. So for every $z \in \mathbb{C} \setminus \sigma (x_0)$ we have

$$\varphi (\left[ ze - x_0 \right]^{-1}) = \frac{1}{\varphi (ze - x_0)} = \frac{1}{(z - z_0) \varphi (z) - \varphi (y_0)} = \frac{1}{z - z_0}.$$

We can now apply to $\varphi$ in $\mathcal{U}$ the Hahn-Banach theorem, $\mathcal{U}$ being locally convex. Thus, let $\tilde{\varphi} : \mathcal{U} \to \mathbb{C}$ be a continuous linear (but not necessarily multiplicative) functional that extends $\varphi$; and set $\psi : D \to \mathcal{U}$ by $\psi (\lambda, z) = \left[ ze - f (\lambda) \right]^{-1}$ for every $(\lambda, z) \in D$. If we prove that $b = \tilde{\varphi} \circ \psi : D \to \mathbb{C}$ is holomorphic, then we shall apply the criterion given by [8, p. 14, Lemma 2.] to conclude that each connected component of $D$ is a domain of holomorphy.

Let $J : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ be the Jordan product $J (x, y) = \frac{x \cdot y + y \cdot x}{2}$, for any $x, y \in \mathcal{U}$ having continuous quasi-inversion, by [10, p. 1686, Proposition 1.] $J$ is jointly continuous. Thus if $(\lambda, z) \in D$ the following limit exists:

$$\frac{\partial \psi}{\partial z} (\lambda, z) = \lim \frac{\psi (\lambda, z + h) - \psi (\lambda, z)}{h} = \lim \frac{[(z + h) e - f (\lambda)]^{-1} - [ze - f (\lambda)]^{-1}}{h} = - \lim J \left( [(z + h) e - f (\lambda)]^{-1}, [ze - f (\lambda)]^{-1} \right) = - J \left( [ze - f (\lambda)]^{-1}, [ze - f (\lambda)]^{-1} \right) = - [ze - f (\lambda)]^{-2}$$
(we have used the equality $x^{-1} - y^{-1} = \frac{x^{-1} \cdot (y - x) \cdot y^{-1} + y^{-1} \cdot (y - x) \cdot x^{-1}}{2}$, true for any $x, y \in \mathcal{U}^{-1}$).

To prove the holomorphicity of $\psi$ in the variable $\lambda$, we need to extend $J$ to $\tilde{J} : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ in a jointly continuous fashion. Indeed, for every $y \in \mathcal{U}$ the map $J_y : \mathcal{U} \to \mathcal{U}$ given by $J_y (x) = J (x, y)$ is continuous and linear, therefore it extends to a continuous linear map $\tilde{J}_y : \mathcal{U} \to \mathcal{U}$: set $\tilde{J} (x, y) = \tilde{J}_y (x)$ for any $(x, y) \in \mathcal{U} \times \mathcal{U}$. Let now $\bar{U}$ be a closed 0-neighbourhood in $\mathcal{U}$ (a topological vector space always admits a fundamental system of 0-neighbourhoods, cfr. [6, p. 16, 1.3.]): then $U = \bar{U} \cap \mathcal{U}$ is a 0-neighbourhood in $\mathcal{U}$, thus a 0-neighbourhood $V$ in $\mathcal{U}$ exists so that $J (V \times V) \subseteq U$. If $\bar{V}$ is the closure of $V$ in $\mathcal{U}$, then $\bar{V}$ is a 0-neighbourhood in $\mathcal{U}$ (cfr. [6, p. 17, 1.5.]), and $\tilde{J} (\bar{V} \times V) \subseteq \bar{U}$: in fact, if $y \in V$, then $J_y (V) \subseteq U$, whence $\tilde{J}_y (V) \subseteq \bar{U}$.

Now let $(\lambda, z) \in D$, and set $g : B (0, \delta) \to \mathcal{U}^{-1} : g (h) = ze - f (\lambda + h)$ ($\delta > 0$ small enough): $g$ is holomorphic in $0$. If $a (h) = \frac{g (h) - g (0)}{h}$ for every $h \in B (0, \delta)$, then an easy computation leads to:

$$
g (h)^{-1} - g (0)^{-1} = \frac{g (h)^{-1} \cdot a (h) \cdot g (0)^{-1} + g (0)^{-1} \cdot a (h) \cdot g (h)^{-1}}{h}
$$

$$
= J \left( \frac{a (h)}{J (g (h)^{-1}, g (0)^{-1})} - J \left( \frac{a (h)}{J (g (h)^{-1}, g (0)^{-1})} \right), g (0)^{-1} \right) -
$$

$$
- J \left( J \left( \frac{a (h)}{J (g (h)^{-1}, g (0)^{-1})} \right), g (h)^{-1} \right).
$$

Since $\lim_{h \to 0} g (h)^{-1} = g (0)^{-1} \in \mathcal{U}$, while $\lim_{h \to 0} a (h) = g' (0) \in \mathcal{U}$, the following limit exists in $\mathcal{U}$:

$$
\frac{\partial \psi}{\partial \lambda} (\lambda, z) = \lim_{h \to 0} \frac{g (h)^{-1} - g (0)^{-1}}{h} = J \left( g' (0), g (0)^{-2} \right) -
$$

$$
- 2 \tilde{J} \left( J \left( g' (0), g (0)^{-1} \right), g (0)^{-1} \right)
$$

(if we could expand $\tilde{J}$, the latter expression would of course equal $-g (0)^{-1} \cdot g' (0) \cdot g (0)^{-1}$).

Therefore, $b$ is separately holomorphic in $D$: (it being continuous on $D$) $b$ is then (jointly) holomorphic in $D$.

Theorem 6. has several consequences. Among them are the logarithmic pluri-sub-harmonicity of several functions of the spectrum in $\mathcal{U}$, such as the spectral radius, any $k$-th spectral diameter (with $k \in \mathbb{N}$), the spectral capacity, and many others. Also, we have: the pluri-analyticity of isolated eigenvalues,
and, more generally, of spectral sets; the finite scarcity and countable scarcity theorems; and so on. For deeper analyses of the consequences of the Oka-analyticity, see e.g. [1], [9], [11], and the literature cited there.

BIBLIOGRAPHY


