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# Partial Hölder continuity for quasilinear parabolic systems of higher order with strictly controlled growth

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**Equazioni a derivate parziali.** — *Partial Hölder continuity for quasilinear parabolic systems of higher order with strictly controlled growth* (*). Nota di MARIO MARINO e ANTONINO MAUGERI, presentata (**) dal Socio G. FICHERA.

RIASSUNTO. — Sfruttando i risultati di [1], si prova che le derivate spaziali $D^{\alpha}u$ di ordine $|\alpha|$ con $|\alpha| < m - 1$ delle soluzioni in Q di un sistema parabolico quasilineare di ordine $2m$ con andamenti strettamente controllati, sono parzialmente hölderiane in Q con esponente di hölderianità decrescente al crescere di $|\alpha|$.

## 1. INTRODUCTION

In [1] we have shown that the "spatial" derivatives of order $m-1$ of the solutions in $Q = \Omega \times (-T, 0)$ ($\Omega$ a bounded open set of $R^n$ $(n > 2)$ with points $x = (x_1, x_2, \ldots, x_n)$; $0 < T < +\infty$) of the quasilinear parabolic system of order $2m$ $(m \geq 1)$

$$(1.1) \qquad (-1)^m \sum_{|\alpha|=m} \sum_{|\beta|=m} D^{\alpha}(A_{\alpha\beta}(X, \delta u) D^{\beta} u) + \frac{\partial u}{\partial t} =$$

$$= (-1)^m \sum_{|\alpha|=m} D^{\alpha} f^{\alpha}(X, \delta u) + \sum_{|\alpha| \leq m-1} (-1)^{|\alpha|} D^{\alpha} f^{\alpha}(X, Du),$$

(here $u : Q \to R^N$ $(N \geq 1)$, $\delta u = \{D^{\alpha}u\}_{|\alpha| \leq m-1}$ (1), $Du = \{D^{\alpha} u\}_{|\alpha| \leq m}$ and $X = (x, t) \in R^{n+1}$) are partially Hölder continuous. In this paper we complete the above result by proving that also the "spatial" derivatives of order $|\alpha|$ with $|\alpha| < m - 1$ $(m > 1)$ are partially Hölder continuous, with a greater Hölder exponent. This result cannot be deduced from the partial Hölder continuity of the derivatives $D^{\alpha}u$ with $|\alpha| = m - 1$ since we have no informations about the derivatives with respect to variable $t$. We are able to deduce it by using the technique of regularization in the spaces $\mathscr{L}^{(2,\theta)}(Q, R^N)$.

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(1) If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multi-index, we set, as usual,

$$|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n, \; D^{\alpha} = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n}, \; D_i = \frac{\partial}{\partial x_i}.$$

Let us recall the hypotheses and the main results of which we shall make use.

We call solution of system (1.1) any vector $u \in L^2(-T, 0, H^m(\Omega, R^N)) \cap \cap L^\infty(-T, 0, L^2(\Omega, R^N))$ such that

$$\int_Q \Big\{ \sum_{|\alpha|=m} \sum_{|\beta|=m} (A_{\alpha\beta} D^\beta u \mid D^\alpha \varphi) - (u \mid \frac{\partial \varphi}{\partial t}) \Big\} dX = \int_Q \sum_{|\alpha| \leq m} (f^\alpha \mid D^\alpha \varphi) \, dX \ (2),$$

$\forall \varphi \in L^2(-T, 0, H_0^m(\Omega, R^N)) \cap H^1(-T, 0, L^2(\Omega, R^N)) : \varphi(x, -T) = = \varphi(x, 0) = 0$ in $\Omega$.

We suppose that the N-vectors $f^\alpha(X, \delta u)$, $|\alpha| = m$, and $f^\alpha(X, Du)$, $|\alpha| \leq m-1$, are measurable in $X \in Q$ and continuous in $\delta u$ and $Du$ respectively, and have the following strictly controlled growth

$$\|f^\alpha(X, \delta u)\| \leq g^\alpha(X) + c \sum_{|\beta| \leq m-1} \|D^\beta u\|^{\theta(m, |\beta|)} \quad , \quad |\alpha| = m,$$

(1.2)

$$\|f^\alpha(X, Du)\| \leq g^\alpha(X) + c \sum_{|\beta| \leq m} \|D^\beta u\|^{\theta(|\alpha|, |\beta|)} \quad , \quad |\alpha| \leq m-1,$$

with

$$1 \leq \theta(m, |\beta|) < \frac{n + 2m}{n + 2|\beta|} \quad , \quad |\beta| \leq m-1,$$

(1.3)

$$1 \leq \theta(|\alpha|, |\beta|) < \frac{n + 4m - 2|\alpha|}{n + 2|\beta|} \quad , \quad |\alpha| \leq m-1, |\beta| \leq m,$$

and

$$g^\alpha(X) \in L^2(Q), \ |\alpha| = m,$$

$$g^\alpha(X) \in L^{2/\bar{\gamma}_{|\alpha|}}(Q), \ |\alpha| \leq m-1, \ \bar{\gamma}_{|\alpha|} = \frac{n + 4m - 2|\alpha|}{n + 2m}.$$

We also assume that $A_{\alpha\beta}(X, p^*)$, $|\alpha| = |\beta| = m$, are $N \times N$ matrices, uniformly continuous and bounded in $\bar{Q} \times \mathscr{R}^*$ (3) and such that

(1.4) $$\sum_{|\alpha|=m} \sum_{|\beta|=m} (A_{\alpha\beta}(X, p^*) \xi^\beta \mid \xi^\alpha) \geq \nu \sum_{|\alpha|=m} \|\xi^\alpha\|^2, \ \nu > 0,$$

for every $(X, p^*) \in \bar{Q} \times \mathscr{R}^*$ and every system $\{\xi^\alpha\}_{|\alpha|=m}$ of vectors of $R^N$.

(2) $(|)_k$ and $\|\cdot\|_k$ are the scalar product and the norm in $R^k$. We omit the index $k$ wherever there is no ambiguity.

(3) $\mathscr{R}^*$ denotes the cartesian product $\prod_{|\alpha| \leq m-1} R_\alpha^N$ with points $p^* = \{p^\alpha\}_{|\alpha| \leq m-1}$, $p^\alpha \in R^N$.

The result proved in [1] (see Theorem 6.I of [1]) is the following

THEOREM 1.1. *Let* $u \in L^2(-T, 0, H^m(\Omega, R^N)) \cap L^\infty(-T, 0, L^2(\Omega, R^N))$ *be a solution of system* (1.1). *Let hypotheses* (1.2), (1.3), (1.4) *be satisfied and moreover let*

$$g^\alpha(X) \in L^p(Q) \quad , \quad |\alpha| = m ,$$

(1.5)

$$g^\alpha(X) \in L^{p/\gamma_{|\alpha|}}(Q) \quad , \quad |\alpha| \leq m - 1 ,$$

*with* $p > n + 2m$. *Then there exists a set* $Q_0 \subset Q$, *closed in* Q, *such that*

$$D^\alpha u \in C^{0,\gamma}(Q \setminus Q_0, R^N) , \forall \gamma < 1 - \frac{n + 2m}{p} , |\alpha| = m - 1 ,$$

*and*

$$\mathcal{M}_{n+2m-2}(Q_0) = 0 ,$$

*where* $\mathcal{M}_{n+2m-2}$ *is the* $(n + 2m - 2)$*-dimensional Hausdorff measure with respect to the parabolic metric*

$$\delta(X, Y) = \max\left\{ \|x - y\| , |t - \tau|^{\frac{1}{2m}} \right\}, X = (x, t) , Y = (y, \tau) \text{ (4)} .$$

We achieved Theorem 1.1 by using some preliminary results that we are going to recall. First, let us give some more notations.

We set

$$\xi = (n + 2m)\left(1 - \frac{2}{p}\right) , \quad p > n + 2m,$$

$$\Phi(X^0, \sigma) = \sigma^\xi + \int_{Q(X^0,\sigma)} \left( \sum_{|\beta| \leq m} \| D^\beta u \|^{\frac{2(n+2m)}{n+2|\beta|}} + \sigma^{-2m} \left\| u - P_{X^0,\sigma} \right\|^2 \right) dX ,$$

$$Q_0 = \left\{ X \in Q : \liminf_{\sigma \to 0} \sigma^{-(n+2m-2)} \Phi(X, \sigma) > 0 \right\} \text{ (5)} ,$$

where $X^0 = (x^0, t^0) , \sigma > 0 , Q(X^0, \sigma) = B(x^0, \sigma) \times (t^0 - \sigma^{2m}, t^0)$

(4) Also the Hölder continuity is related to this metric.

(5) We can show that if the assumptions of Theorem 1.1 are satisfied, then $\mathcal{M}_{n+2m-2}(Q_0) = 0$.

($B(x^0, \sigma) = \{x \in R^n : \| x - x^0 \| < \sigma\}$) and $P_{X^0,\sigma}$ is the vector polynomial in $x$, of degree at most $m-1$, such that:

$$\int_{Q(X^0,\sigma)} D^\alpha \left( u - P_{X^0,\sigma} \right) dX = 0 \quad , \quad \forall \alpha, |\alpha| \leq m-1 .$$

Then a first result is the following:

LEMMA 1.1. *If* $u \in L^2(-T, 0, H^m(\Omega, R^N)) \cap L^\infty(-T, 0, L^2(\Omega, R^N))$ *is a solution of system* (1.1) *and the hypotheses* (1.2), (1.3), (1.4) *and* (1.5) *are fulfilled, then,* $\forall X^0 \in Q \setminus Q_0$ *and* $\forall \lambda \in (n + 2m - 2, \xi)$, *there exist* $\sigma_\lambda < 1$ *and* $r > 0$ *with* $Q(X^0, r + \sigma_\lambda) \subset Q$ *such that,* $\forall Y \in Q(X^0, r)$ *and* $\forall \tau \in (0, 1)$,

$$\Phi(Y, \tau\sigma_\lambda) \leq c\tau^\lambda \Phi(Y, \sigma_\lambda) \text{ (6)} . \tag{1.6}$$

Moreover, the following lemmas hold:

LEMMA 1.2 (POINCARÈ). *Let* $u \in L^2(t^0 - \sigma^{2m}, t^0, H^m(B(x^0, \sigma), R^N)) \cap$ $\cap H^{1/2}(t^0 - \sigma^{2m}, t^0, L^2(B(x^0, \sigma), R^N))$, *then the following inequality holds*:

$$\sigma^{-2m} \int_{Q(X^0,\sigma)} \left\| u - P_{X^0,\sigma} \right\|^2 dX \leq \tag{1.7}$$

$$\leq c \left\{ \int_{Q(X^0,\sigma)} \sum_{|\alpha| = m} \| D^\alpha u \|^2 dX + \right.$$

$$\left. + \int_{t^0 - \sigma^{2m}}^{t^0} dt \int_{t^0 - \sigma^{2m}}^{t^0} d\xi \int_{B(x^0,\sigma)} \frac{\| u(x, t) - u(x, \xi) \|^2}{|t - \xi|^2} dx \right\} \text{ (7)} .$$

LEMMA 1.3 (EHRLING-NIRENBERG-GAGLIARDO). *Let* $u \in L^2(t^0 - \sigma^{2m}, t^0,$ $H^m(B(x^0, \sigma), R^N))$, *then, for every* $\alpha$, $|\alpha| \leq m-1$, *the following inequality holds :*

$$\int_{Q(X^0,\sigma)} \left\| D^\alpha \left( u - P_{X^0,\sigma} \right) \right\|^2 dX \leq \tag{1.8}$$

$$\leq c\sigma^{2m-2|\alpha|} \int_{Q(X^0,\sigma)} \left( \sum_{|\beta| = m} \| D^\beta u \|^2 + \sigma^{-2m} \left\| u - P_{X^0,\sigma} \right\|^2 \right) dX .$$

Finally, we can state the result of this paper.

(6) See [1], n. 6.
(7) See [2], Lemma 2.I.

Theorem 1.2. *Let the assumptions of Theorem* 1.1 *be satisfied. Then, for every* $\alpha$, $|\alpha| < m-1$ $(m>1)$, *we have*

$$D^\alpha u \in C^{0,\gamma}(Q \setminus Q_0, R^N) \quad , \quad \forall \gamma < 1 - \frac{n+2|\alpha|+2}{p}.$$

## 2. Proof of Theorem 1.2.

Let $\alpha$ be a multi-index with $|\alpha| < m-1$ $(m>1)$, then, for every $Q(Y,\sigma) \Subset Q$ (8), $\sigma \leq 1$, we have (9)

$$(2.1) \qquad \int_{Q(Y,\sigma)} \| D^\alpha u - (D^\alpha u)_{Q(Y,\sigma)} \|^2 dX \leq$$

$$\leq 2 \int_{Q(Y,\sigma)} \| D^\alpha (u - P_{Y,\sigma}) \|^2 dX + 2 \int_{Q(Y,\sigma)} \| D^\alpha P_{Y,\sigma} - (D^\alpha u)_{Q(Y,\sigma)} \|^2 dX.$$

Now, taking into account the estimate

$$\| D^\alpha P_{Y,\sigma} - (D^\alpha u)_{Q(Y,\sigma)} \| \leq c \sum_{|\alpha| < |\beta| \leq m-1} \| (D^\beta u)_{Q(Y,\sigma)} \| \, \sigma^{|\beta| - |\alpha|}$$

and the inequality (1.8), we obtain from (2.1):

$$(2.2) \qquad \int_{Q(Y,\sigma)} \| D^\alpha u - (D^\alpha u)_{Q(Y,\sigma)} \|^2 dX \leq$$

$$\leq c\sigma^{2m-2|\alpha|} \int_{Q(Y,\sigma)} \left( \sum_{|\beta|=m} \| D^\beta u \|^2 + \sigma^{-2m} \| u - P_{Y,\sigma} \|^2 \right) dX +$$

$$+ c \sum_{|\alpha| < |\beta| \leq m-1} \sigma^{2|\beta| - 2|\alpha| + n + 2m} \| (D^\beta u)_{Q(Y,\sigma)} \|^2 \leq$$

$$\leq c\sigma^{2m-2|\alpha|} \Phi(Y,\sigma) + c \sum_{|\alpha| < |\beta| \leq m-1} \sigma^{2|\beta| - 2|\alpha| - n - 2m} \left( \int_{Q(Y,\sigma)} \| D^\beta u \| dX \right)^2.$$

By observing that

$$\int_{Q(Y,\sigma)} \| D^\beta u \| dX \leq c\sigma^{(n+2m)\left(1 - \frac{n+2|\beta|}{2(n+2m)}\right)} \left( \int_{Q(Y,\sigma)} \| D^\beta u \|^{\frac{2(n+2m)}{n+2|\beta|}} dX \right)^{\frac{n+2|\beta|}{2(n+2m)}}$$

(8) We say that $Q(Y,\sigma) \Subset Q(Y = (y,\tau))$ if $B(y,\sigma) \Subset \Omega$ and $\sigma^{2m} < \tau + T \leq T$.

(9) In the following we set

$$(D^\alpha u)_{Q(Y,\sigma)} = \frac{1}{meas\, Q(Y,\sigma)} \int_{Q(Y,\sigma)} D^\alpha u \, dX.$$

it follows

$$\sigma^{2|\beta|-2|\alpha|-n-2m}\left(\int_{Q(Y,\sigma)} \| D^\beta u \| \, dX\right)^2 \leq \tag{2.3}$$

$$\leq c\sigma^{2m-2|\alpha|+\xi\frac{n+2|\beta|}{n+2m}}\left(\sigma^{-\xi}\int_{Q(Y,\sigma)} \| D^\beta u \|^{\frac{2(n+2m)}{n+2|\beta|}} dX\right)^{\frac{n+2|\beta|}{n+2m}} \leq$$

$$\leq c\sigma^{2m-2|\alpha|+\xi\frac{n+2|\beta|}{n+2m}}\left(1+\sigma^{-\xi}\int_{Q(Y,\sigma)} \| D^\beta u \|^{\frac{2(n+2m)}{n+2|\beta|}} dX\right) \leq$$

$$\leq c\sigma^{2m-2|\alpha|+\xi\frac{n+2|\beta|}{n+2m}-\xi}\,\Phi(Y,\sigma)\,.$$

From (2.2) and (2.3) we deduce:

$$\int_{Q(Y,\sigma)} \| D^\alpha u - (D^\alpha u)_{Q(Y,\sigma)} \|^2 \, dX \leq$$

$$\leq c\sigma^{2m-2|\alpha|}\,\Phi(Y,\sigma) + c \sum_{|\alpha|<|\beta|\leq m-1} \sigma^{2m-2|\alpha|-(m-|\beta|)(2-4/p)}\,\Phi(Y,\sigma)$$

and since

$$2m-2|\alpha| \geq 2+\frac{4}{p}(m-|\alpha|-1)\,,$$

$$2m-2|\alpha|-(m-|\beta|)\left(2-\frac{4}{p}\right) \geq 2+\frac{4}{p}(m-|\alpha|-1)\,,$$

it results

$$\int_{Q(Y,\sigma)} \| D^\alpha u - (D^\alpha u)_{Q(Y,\sigma)} \|^2 \, dX \leq c\sigma^{2+(4/p)(m-|\alpha|-1)}\,\Phi(Y,\sigma)\,. \tag{2.4}$$

Let us now write (2.4) for $Y \in Q(X^0, r)$, $\sigma = \tau\sigma_\lambda$, $\tau \in (0,1)$, $\lambda \in (n+2m-2, \xi)$ ($X^0$, $\sigma_\lambda$, $r$ are the same as in Lemma 1.1). We have

$$\int_{Q(Y,\tau\sigma_\lambda)} \| D^\alpha u - (D^\alpha u)_{Q(Y,\tau\sigma_\lambda)} \|^2 \, dX \leq c\tau^{2+(4/p)(m-|\alpha|-1)}\,\Phi(Y,\tau\sigma_\lambda)$$

and from (1.6)

$$\int_{Q(Y,\tau\sigma_\lambda)} \| D^\alpha u - (D^\alpha u)_{Q(Y,\tau\sigma_\lambda)} \|^2 \, dX \leq c\tau^{2+\lambda+(4/p)(m-|\alpha|-1)}\,\Phi(Y,\sigma_\lambda)\,. \tag{2.5}$$

Taking into account Poincaré's inequality (1.7), it follows from (2.5)

$$\text{(2.6)} \qquad \int_{Q(Y,\tau\sigma\lambda)} \| D^\alpha u - (D^\alpha u)_{Q(Y,\tau\sigma\lambda)} \|^2 \, dX \leq$$

$$\leq K(u)\, \tau^{2+\lambda+(4/p)(m-|\alpha|-1)} \quad , \qquad \forall\, Y \in Q(X^0, r) \,, \forall\, \tau \in (0\,,1) \,.$$

Since

$$2 + \lambda + \frac{4}{p}(m - |\alpha| - 1) > n + 2m \,,$$

the inequality (2.6) ensures that

$$D^\alpha u \in C^{0,\gamma}(\overline{Q(X^0, r)}, R^N) \quad , \qquad \forall \gamma < 1 - \frac{n + 2|\alpha| + 2}{p} \,.$$

Thus we have proved that, for every $\alpha$, $|\alpha| < m - 1$ $(m > 1)$

$$D^\alpha u \in C^{0,\gamma}(Q \setminus Q_0 \,, R^N) \quad , \qquad \forall \gamma < 1 - \frac{n + 2|\alpha| + 2}{p} \,.$$

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