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A note on the minimal normal Fitting class

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RENDICONTI

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Classe di Scienze fisiche, matematiche e naturali

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Presiede il Presidente della Classe Giuseppe Montalenti

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Teoria dei gruppi. — *A note on the minimal normal Fitting class.* Nota di Marco Barlotti, presentata (*) dal Socio G. Zappa.

Riassunto. — Un gruppo finito ciclico-per-nilpotente appartiene alla minima classe di Fitting normale se e solo se è nilpotente.

1. Introduction

All the groups considered in this paper are supposed to be finite and soluble. A *Fitting pair* (see [6]) is a pair (A, d) where A is an abelian group and d assigns to each group G a homomorphism d_G of G into A such that (1) whenever G, H are groups and $\alpha : G \rightarrow H$ is a normal embedding, $d_G = \alpha d_H$ and (2) for every $\boldsymbol{a} \in A$ there exist a group G and a $\boldsymbol{g} \in G$ such that $\boldsymbol{g} d_G = \boldsymbol{a}$. The class of all the group G such that $G d_G = 1$ is called the *kernel* of the pair (A, d) and is a normal Fitting class.

Blessenhol and Gaschutz introduced in [1] Fitting pairs built upon permutation representations and upon the determinants of the linear maps induced by conjugacy on certain chief factors; these constructions have been afterwards

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generalized by many authors. Transfer Fitting pairs were introduced by Laue, Lausch and Pain in [5] and have been used in a more elaborate version by Berger ([2]) to give a description of the unique minimal normal Fitting class $\mathfrak{S}_*$. We use a special case of the transfer Fitting pairs to prove that a cyclic-by-nilpotent group (i.e., a group G which possesses a cyclic normal subgroup N such that G/N is nilpotent) is in $\mathfrak{S}_*$ if and only if it is nilpotent; note that the group $\langle x, y / x^{25} = 1, y^2 = 1, yxy = x^{-1} \rangle$ seems to belong to the kernel of all the possible Fitting pairs defined in the track of [1]. Then we give a construction which shows that some generalizations of the main theorem are not possible.

The notation is conventional: see, e.g., [3] and [4]. The basic information on normal Fitting classes can be found in [1] and in [3].

2. The Fitting pair $(\mathrm{S}, \mathrm{d}^{p,q})$

Let p be a prime, and let q be a prime dividing $p - 1$; let Z_p be the cyclic group of order p, and let S be the Sylow q-subgroup of $\mathrm{Aut}(\mathrm{Z}_p)$. We now assign to every group G a homomorphism $\mathrm{d}_{\mathrm{G}}^{p,q} : \mathrm{G} \to \mathrm{S}$. Since we are describing a special case of the Fitting pairs defined by Berger, for the proof that all the mappings involved are well defined and that $(\mathrm{S}, \mathrm{d}^{p,q})$ fulfils the requirements for a Fitting pair the reader is referred to [2] or, better, to [3].

Let G be a group, let X be a subnormal subgroup of G of order p and let N be the normalizer of X in G. Choose an isomorphism $\psi : \mathrm{X} \to \mathrm{Z}_p$, and define as follows an homomorphism $\tilde{\psi} : \mathrm{N} \to \mathrm{Aut}(\mathrm{Z}_p)$: for any $y \in \mathrm{N}$, $\boldsymbol{y}\tilde{\psi}$ is the automorphism of Z_p which maps the element $\boldsymbol{z}$ to $(\boldsymbol{y}^{-1}(\boldsymbol{z}\psi^{-1})\boldsymbol{y})\psi$. Now choose a Sylow q-subgroup Q of N, and denote by $v_{\mathrm{G}\to\mathrm{Q}}$ the transfer of G into Q/Q'. We define a mapping $\vartheta_{\mathrm{X}} : \mathrm{G} \to \mathrm{S}$ thus: if $\boldsymbol{g} \in \mathrm{G}$ and $\boldsymbol{g}v_{\mathrm{G}\to\mathrm{Q}} = \mathrm{Q}'\boldsymbol{y}$, then $\boldsymbol{g}\vartheta_{\mathrm{X}} = \boldsymbol{y}\tilde{\psi}$. Finally, let q^e be the exponent of S and let $t(\mathrm{X})$ be an integer such that $t(\mathrm{X}) \mid \mathrm{N} : \mathrm{Q} \mid \equiv 1 \pmod{q^e}$.

We now define the homomorphism $\mathrm{d}_{\mathrm{G}}^{p,q}$. If G has no subnormal subgroups of order p, then $\mathrm{d}_{\mathrm{G}}^{p,q}$ maps G onto the identity subgroup of G. Otherwise, let $[\mathrm{X}_1], \ldots, [\mathrm{X}_k]$ be the distinct conjugacy classes which make up the set of the subnormal subgroups of G of order p; for any $\boldsymbol{g} \in \mathrm{G}$ we define $\boldsymbol{g}\mathrm{d}_{\mathrm{G}}^{p,q} =$

$$= \prod_{i=1}^{k} (\boldsymbol{g}\vartheta_{\mathrm{X}_i})^{t(\mathrm{X}_i)} .$$

3. Proof of the theorem

Theorem *Let* G *be a cyclic-by-nilpotent group which is not nilpotent. Then* $\mathrm{G} \notin \mathfrak{S}_*$.

Proof. Let G be a minimal counter-example; then there exists an element $\boldsymbol{x}$ of G such that $\langle \boldsymbol{x} \rangle \lhd \mathrm{G}$ and $\mathrm{G}/\langle \boldsymbol{x} \rangle$ is nilpotent, G is not nilpotent and G belongs to $\mathfrak{S}_*$.

Take a chief series of G to which $\langle x \rangle$ belongs; since G is not nilpotent, there exist a chief factor H/K of this series and an element y of G such that y does not centralize H/K; and, since $G/\langle x \rangle$ is nilpotent, it must be $H = \langle x^i \rangle$ for a certain positive integer i (and $K = \langle x^{ip} \rangle$ for a certain prime p). Write the period of x as iph, and let $X = \langle x^{ih} \rangle$; X is a normal subgroup of G of order p. It must be $y^{-1} x^i y = x^{im}$ with $m \not\equiv 1 \pmod p$, whence $y^{-1} x^{ih} y = y^{-1} (x^i)^h y = (x^{im})^h = (x^{ih})^m \neq x^{ih}$ and we have proved that y does not centralize X. Now write y as a product of elements whose order is a power of a prime: these elements cannot all centralize X, so there exists an element y_1 of G, whose order is a power of a certain prime q, which does not centralize X; since y_1 induces a q-automorphism in X, q divides $p-1$ (and, in particular, $q \neq p$).

We want to prove that $y_1 \, d_G^{p,q} \neq 1$, whence G does not belong to the kernel of the Fitting pair $(S, d^{p,q})$ defined in section 2, and this contradicts the assumption that $G \in \mathfrak{S}_*$.

The subgroup $\langle x, y_1 \rangle$ is not nilpotent; it is clearly cyclic-by-nilpotent, and it belongs to $\mathfrak{S}_*$ because it is subnormal in G (by the nilpotency of $G/\langle x \rangle$) and $G \in \mathfrak{S}_*$. So by the minimality of G we must have $G = \langle x, y_1 \rangle$: this yields that X is the unique subgroup of G which has order p, hence to prove that $y_1 \, d_G^{p,q} \neq 1$ we only have to show that $y_1 \vartheta_X \neq 1$ or, which is equivalent, that if Q is the chosen Sylow q-subgroup of G and $y_1 v_{G \to Q} = Q' g$ then g does not centralize X.

Since Q can be any Sylow q-subgroup of G, we choose it to contain y_1. Let $\{t_1, \dots, t_w\}$ be a complete set of right coset representatives of Q in G; since $G = \langle x, y_1 \rangle$ with $\langle x \rangle \lhd G$, each t_i can be written as $a_i x_i$ where $a_i \in \langle y_1 \rangle \leq Q$ and $x_i \in \langle x \rangle$ $(1 \leq i \leq w)$: hence $\{x_1, \dots, x_w\}$ is a complete set of right coset representatives of Q in G all of whose elements belong to $\langle x \rangle$. For every $i \in \{1, \dots, w\}$ there exist a $g_i \in Q$ and a $j(i) \in \{1, \dots, w\}$ such that $x_i y_1 = g_i x_{j(i)}$ and by definition of the transfer homomorphism we have

$$y_1 v_{G \to Q} = Q' \prod_{i=1}^{w} g_i = Q' \prod_{i=1}^{w} (x_i y_1 x_{j(i)}^{-1}).$$

Let $g = \prod_{i=1}^{w} (x_i y_1 x_{j(i)}^{-1})$. Clearly g acts on X (by conjugacy) in exactly the same way as y_1^w does; but, since w is prime with q, $\langle y_1^w \rangle = \langle y_1 \rangle$ whence y_1^w induces on X a non-trivial automorphism, and so does g.

COROLLARY. *The $\mathfrak{S}_*$-radical of a cyclic-by-nilpotent group is its Fitting subgroup.*

Proof. By the previous theorem, the $\mathfrak{S}_*$-radical of a cyclic-by-nilpotent subgroup (being itself cyclic-by-nilpotent) is nilpotent, hence contained in the Fitting subgroup. Since the reverse inclusion is true for every group, the corollary is proved.

4. Final remarks

We conclude with a result which limits the possible generalizations of the previous theorem.

THEOREM. *Let* G *be a group. Suppose that there exist subgroups* N_0, N, A *of* G *such that*

(*a*) $G = N_0 NA$;

(*b*) $N_0, N \lhd G$;

(*c*) $N_0, N \in \mathfrak{S}_*$ *and* A *is abelian*;

(*d*) $N_0 \cap N = A \cap N = A \cap N_0 = 1$;

(*e*) *there exists an isomorphism* $\varphi : N_0 \rightarrow N$ *such that for any* $\boldsymbol{a} \in A$ *and for any* $\boldsymbol{n}_0 \in N_0$ $(\boldsymbol{a}^{-1} \boldsymbol{n}_0 \boldsymbol{a}) \varphi = \boldsymbol{a} (\boldsymbol{n}_0 \varphi) \boldsymbol{a}^{-1}$.

Then $G \in \mathfrak{S}_*$.

Proof. We give a sketch of the proof, leaving the details to the reader. Let D be the direct product of two isomorphic copies of NA, and let α, β be isomorphisms which map NA onto the direct factors of D; then D is the internal direct product of $(NA)\,\alpha$ and $(NA)\,\beta$. Note that $N\alpha$ and $N\beta$ are normal subgroups of D, whence by (*c*) they are contained in the $\mathfrak{S}_*$-radical $D_{\mathfrak{S}_*}$ of D. Let $H = \{\boldsymbol{d} \in D / \boldsymbol{d} = (\boldsymbol{a}\,\alpha)^{-1} (\boldsymbol{a}\,\beta)$ with $\boldsymbol{a} \in A\}$; by Lemma 2.3 of [6], $H \leq D_{\mathfrak{S}_*}$. Now let $K = (N\alpha)(N\beta)H$; clearly $K \leq D_{\mathfrak{S}_*}$ and (since A is abelian) $K \lhd D$, whence $K \in \mathfrak{S}_*$. Finally, for any $\boldsymbol{n}_0 \in N_0$ put $\boldsymbol{n}_0 \eta = \boldsymbol{n}_0 \varphi\alpha$; for any $\boldsymbol{n} \in N$ put $\boldsymbol{n}\,\eta = \boldsymbol{n}\,\beta$; and for any $\boldsymbol{a} \in A$ put $\boldsymbol{a}\,\eta = (\boldsymbol{a}\,\alpha)^{-1} (\boldsymbol{a}\,\beta)$. We have thus defined a map η from $N_0 \cup N \cup A$ to K which extends to a homomorphism $\tilde{\eta}$ of G onto K; since G and K are easily seen to have the same order, $\tilde{\eta}$ is in fact an isomorphism and we have proved that $G \in \mathfrak{S}_*$.

To obtain groups which satisfy the hypotheses of this theorem, take any group N in $\mathfrak{S}_*$ and let A be an abelian group such that there exists a non-trivial homomorphism δ of A into Aut (N); let ϑ be the automorphism of A which inverts every element and let N_0 be an isomorphic copy of N. The group we want is the semidirect product of $N \times N_0$ by A with respect to δ for the action of A on N and to $\vartheta\delta$ for the action of A on N_0.

In particular, take N to be cyclic of prime order and A to be cyclic: this example shows that in the condition " cyclic-by-nilpotent " of the theorem in section 3 " cyclic " cannot be replaced by " elementary abelian " nor can " nilpotent " be replaced by " supersoluble ".

REFERENCES

[1] D. BLESSENHOL and W. GASCHÜTZ (1970) – *Über normale Schunck- und Fittingklassen*, « Math. Z », *118*, 1-8.

[2] T.R. BERGER (1981) – *The smallest normal Fitting class revealed*, « Proc. London Math. Soc. » (3), *42* (1), 59–86.

[3] K. DOERK and T.O. HAWKES – *Finite soluble groups*, to appear.

[4] D. GORENSTEIN (1968) – *Finite groups*, Harper & Row.

[5] H. LAUE, H. LAUSCH and G.R. PAIN (1977) – *Verlagerung und normale Fittingklassen endlicher Gruppen*, « Math. Z. », *154*, 257–260.

[6] H. LAUSCH (1973) – *On normal Fitting classes*, « Math. Z. », *130*, 67–72.