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Geometria algebrica. — *Some topological conditions for projective algebraic manifolds with degenerate dual varieties: connections with* **P** *- bundles* (*). Nota di ANTONIO LANTERI (**) e DANIELE STRUPPA (***), presentata (****) dal Socio E. MARCHIONNA.

RIASSUNTO. — Si illustrano alcune relazioni tra le varietà proiettive complesse con duale degenere, le varietà la cui topologia si riflette in quella della sezione iperpiana in misura maggiore dell'ordinario e le varietà fibrate in spazi lineari su di una curva.

INTRODUCTION

This Note deals with three kinds of geometrical objects:

a) projective manifolds whose dual varieties have dimensions less than the ordinary,

b) projective manifolds whose topology reflects those of their hyperplane sections beyond what is prescribed by the first Lefschetz theorem and

c) scrolls over a curve.

Some connections between *b*) and *c*) are discussed in (1.2); (2.1) shows the stronger relations which in dimension three tie *a*), *b*) and *c*). Finally, very simple proofs of two recent results of Ein [1] connecting *a*) and *c*) are provided ((3.3) and (3.4)).

Further relations between *a*) and *b*) are discussed in [6] where this subject is developed in a more systematic way (and to which we refer the reader for the proofs not given here).

The authors are indebted to Prof. S. Kleiman for calling their attention to Ein's preprint [1].

1. Let $X \subset \mathbf{P}^n$ be a complex connected projective algebraic manifold of dimension k and let H be its general hyperplane section. A more or less known result is

(1.1) THEOREM. *If* $k = 2$, *the following facts are equivalent*:

i) $H_1(X, R) \simeq H_1(H, R)$ $(R = \mathbf{Z}$ *or* $\mathbf{Q})$;

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ii) $(X, [H]) = (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(e))$, $e = 1, 2$, *or* X *is a* $\mathbf{P}^1$*-bundle over a smooth curve and* $[H]_{|f} = \mathcal{O}_{\mathbf{P}^1}(1)$ *for any fibre* f *of* X;

iii) $h^{1,0}(X) = h^{1,0}(H)$;

iv) $h^0(K_X \otimes [H]) = 0$, K_X *being the canonical bundle.*

Proof. It is obvious that i) $\leftrightarrow$ iii) and ii) $\rightarrow$ iv). The implication iv) $\rightarrow$ iii) follows from the Kodaira vanishing and the Riemann-Roch theorems, taking into account that $h^{1,0}(H) \geq h^{1,0}(X)$ by the first Lefschetz theorem. For the implication iii) $\rightarrow$ ii) see [8, (1.5.2)].

A natural way for extending (1.1) to higher dimensions is the classical technique of cutting X with hyperplanes until obtaining a surface. We thus obtain

(1.2) THEOREM. *If* $k \geq 3$, *the following facts are equivalent*:

i) $H_1(E, R) \simeq H_1(X, R)$, *where* E *is the general section of* X *with a linear space of* $\mathbf{P}^n$ *of dimension* $n - k + 1$;

ii) *either* $(X, [H]) = (\mathbf{P}^k, \mathcal{O}_{\mathbf{P}^k}(1))$, *or* X *is a hyperquadric of* $\mathbf{P}^{k+1}$, *or* X *is a* $\mathbf{P}^{k-1}$*-bundle over a smooth curve and* $[H]_{|f} = \mathcal{O}_{\mathbf{P}^{k-1}}(1)$ *for any fibre* f *of* X;

iii) $h^0(K_X \otimes [H]^{\otimes(k-1)}) = 0$.

Proof. The equivalence i) $\leftrightarrow$ ii) is proved in [5, (4.2)] (see also [3, (1.1)]). The implication ii) $\rightarrow$ iii) follows from a simple argument based on the adjunction formula; if iii) holds, the Kodaira vanishing theorem and an inductive argument show that the surface section of X satisfies one of the equivalent conditions in (1.1). Then the first Lefschetz theorem yields i).

2. A different approach to generalize (1.1) to higher dimensions consists in considering the pairs (X, L) = (projective manifold, very ample line bundle) with $\dim X = k$ such that

$$H_{k-1}(H, \mathbf{Q}) \simeq H_{k-1}(X, \mathbf{Q}), \ H \text{ a general element of } |L|.$$

Let $\mathcal{L}_k$ be the class of such pairs. In [6] to describe $\mathcal{L}_k$ we introduce the auxiliary class $\mathcal{D}_k$ of the pairs (X, L) as above such that $H_i(H, \mathbf{Q}) \simeq H_i(X, \mathbf{Q})$ for $i \leq 2k - 3$. For $k = 3$, using Picard-Lefschetz theory and a result of Griffiths-Harris [2, (3.26)], we prove [6]

(2.1) THEOREM. *The following facts are equivalent*:

i) $(X, L) \in \mathcal{L}_3$;

ii) $(X, L) \in \mathcal{D}_3$;

iii) $(X, L) = (\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1))$, *or* X *is a* $\mathbf{P}^2$*-bundle over a smooth curve and* $L_{|f} = \mathcal{O}_{\mathbf{P}^2}(1)$ *for any fibre* f *of* X;

iv) *the dual variety of* X *embedded by* L *is degenerate.*

By the expression "X embedded by L" we mean any image $\varphi_V(X)$ of X via the map $\varphi_V : X \to \mathbf{P}(V^*)$ associated to a linear subspace V of $H^0(X, L)$, provided that it is an embedding. X embedded by L has degenerate dual variety (shortly (X, L) has degenerate dual variety) means that the dual variety $(\varphi_V(X))^*$ of $\varphi_V(X)$ has dimension less than the ordinary dim $V - 2$. This fact actually does not depend on the subspace V; to see why, recall that the inclusion $V \subseteq H^0(X, L)$ is strict iff $\varphi_V(X)$ can be realized as a projection from a suitable linear space of $\mathbf{P}(H^0(X, L)^*)$ of the linearly normal model $\varphi_{H^0(L)}(X)$. This is enough in view of

(2.2) *Remark.* Let $Y \subset \mathbf{P}^n$ be a complex connected algebraic manifold and assume that $Y' \subset \mathbf{P}^{n-1}$ is obtained by projecting isomorphically Y from a point $p_0 \in \mathbf{P}^n \setminus Y$. Then $\dim Y'^* = \dim Y^* - 1$. Indeed, the projection from p_0 induces an injection $\mathbf{P}^{n-1*} \to \mathbf{P}^{n*}$ identifying $\mathbf{P}^{n-1*}$ with the hyperplane of $\mathbf{P}^{n*}$ consisting of the hyperplanes of $\mathbf{P}^n$ through p_0 and this exhibits Y'^* as a hyperplane section of Y^*.

Another proof of the equivalence ii) $\leftrightarrow$ iii) in (2.1) can be given, independently on [2], using as the main tool the following proposition whose proof [6] only involves topological arguments.

(2.3) PROPOSITION. *Let* $(X, L) \in \mathscr{D}_k$; *then* (X, L) *has degenerate dual variety iff* $(H, L|_H) \in \mathscr{L}_{k-1}$, H *any general element of* $|L|$.

Finally notice that the analogues of the implications iii) $\to$ ii) $\to$ i) and iii) $\to$ iv) $\to$ i) hold in every dimension $k \geq 3$.

3. Now assume that (X, L) is as in sec. 2. The above argument allows us to define the defect of (X, L) as the integer

$$\operatorname{def}(X, L) = \dim V - 2 - \dim(\varphi_V(X))^*$$

(which of course does not depend on V).

Basic results of Landman and Hefez-Kleiman [4] state that

(3.1) $$\operatorname{def}(X, L) - \dim X \textit{ is even};$$

(3.2) $\operatorname{def}(H, L|_H) = \max\{0, \operatorname{def}(X, L) - 1\}$, H *any general element of* $|L|$.

These results jointly with (1.2), (2.1) allow us to give very short proofs of two theorems recently established by Ein [1, (4.2), and (4.3)] through a careful study of the arrangement of the linear spaces contained in a projective manifold with degenerate dual variety.

(3.3) COROLLARY. *Let* $\dim X = 4$; *the following facts are equivalent*:

i) (X, L) *has degenerate dual variety*;

ii) $(X, L) = (\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(1))$, *or* X *is a* $\mathbf{P}^3$*-bundle over a smooth curve and* $L|_f = \mathcal{O}_{\mathbf{P}^3}(1)$ *for any fibre* f *of* X.

Proof. ii) $\rightarrow$ i). This is obvious if $(X, L) = (\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(1))$. Otherwise X embedded by L is a scroll; hence its general hyperplane section pencil contains no singular section i.e. the corresponding line of the dual projective space does not intersect $(\varphi_{H^0(L)}(X))^*$ which implies i).

i) $\rightarrow$ ii). By (3.1), (3.2) it follows that $\operatorname{def}(H, L|_H) > 0$. Hence by (2.1) either $(H, L|_H) = (\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1))$ or $\varphi_{H^0(L)}(H)$ is a scroll. In the former case it is obvious that $(X, L) = (\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(1))$, while in the latter one the thesis follows from (1.2).

(3.4) COROLLARY. *Assume* $(X, L) \neq (\mathbf{P}^k, \mathcal{O}_{\mathbf{P}^k}(1))$. *Then* $\operatorname{def}(X, L) \leq$ $\leq k - 2$; *for* $k \geq 3$ *equality holds iff* X *is a* $\mathbf{P}^{k-1}$*-bundle over a smooth curve and* $L|_f = \mathcal{O}_{\mathbf{P}^{k-1}}(1)$ *for any fibre* f *of* X.

Proof. Assume $\operatorname{def}(X, L) > k - 2$; then a repeated application of (3.2) shows that $(S, L|_S)$ has degenerate dual variety where S is the general surface section of $\varphi_{H^0(L)}(X)$. But since a smooth surface of degree d has class $\geq d - 1$ [7], it follows that $(S, L|_S) = (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1))$ (see also [2, (3.19)]) and therefore $(X, L) = (\mathbf{P}^k, \mathcal{O}_{\mathbf{P}^k}(1))$, contradiction. Now let $k \geq 3$ and $\operatorname{def}(X, L) = k - 2$. If $k = 3$ there is nothing to prove in view of (2.1). If $k \geq 4$, (3.2) shows that $\operatorname{def}(H, L|_H) = \dim H - 2 > 0$; this is enough to conclude by induction and (1.2). To see the converse use again (1.2) and (3.2).

The above techniques together with topological arguments can be exploited to provide some other results on the defects of suitable classes of projective manifolds. Results, detailed proofs and examples will appear in [6].

Added in proof. After this paper was written, L. Ein kindly sent us a new version of [1]. There he gives another proof of (3.4) which uses the analysis of the arrangement of linear spaces together with adjunction.

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