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Periodic solutions of infinite dimensional Riccati Equations

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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta del 24 novembre 1984

Presiede il Presidente della Classe GIUSEPPE MONTALENTI

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Analisi matematica. — *Periodic solutions of infinite dimensional Riccati Equations.* Nota di GIUSEPPE DA PRATO, presentata (*) dal Corrisp. R. CONTI.

RIASSUNTO. — Si dà un risultato di esistenza di soluzioni periodiche per una equazione di Riccati in dimensione infinita.

1. NOTATION

We denote by H a complex Hilbert space, by $\mathscr{L}(H)$ the Banach algebra of linear bounded operators in H; we set:

$$\Sigma(H) = \{T \in \mathscr{L}(H) \; ; \; T = T^*\} , \qquad \Sigma^+(H) = \{T \in \Sigma(H) \; ; \; T \geq 0\}$$

U will represent the Hilbert space of controls.

Finally for any interval $[a, b]$ we shall denote by $C_s([a, b]; \Sigma^+(H))$ the set of all mappings $[a, b] \to \Sigma^+(H)$, $t \to T(t)$ such that $T(.)x$ is continuous for any $x \in H$.

We shall be concerned with the following Riccati equation:

$$P' = A^*P + PA - PBB^*P + M \tag{1.1}$$

(*) Nella seduta del 24 novembre 1984.

under the following hypotheses:

(1.2)
$$\begin{cases} a) \ A \text{ is the infinitesimal generator of a strongly continuous semigroup } e^{tA} \text{ in } H. \\ b) \ B \in \mathscr{L}(U\,;H) \\ c) \ M \in C_s([0,\rho]\,;\Sigma^+(H)), \ \rho > 0. \end{cases}$$

We shall write a solution of (1.1) under the following integral form:

(1.3)
$$P(t)\,x = e^{tA^*}\,P(0)\,e^{tA}\,x + \int_0^t e^{(t-s)A^*}\,(M(s) - P(s)\,BB^*\,P(s)\,e^{(t-s)A}\,x\,ds\,, \quad x \in H.$$

The following result is proved in [2] and [1].

PROPOSITION 1.1. *Assume* (1.2) *and let* P_0 *belong to* $\Sigma^+(H)$. *Then there exists a unique solution* $P \in C_s([0,\rho]\,;\Sigma^+(H))$ *of* (1.3) *with* $P(0) = P_0$. *Moreover, setting* $P = \wedge(P_0, M)$ *we have:*

(1.4)
$$\wedge(P_0, M) \le \wedge(\bar{P}_0, \bar{M}) \ \text{ if } P_0 \le \bar{P}_0 \text{ and } M \le \bar{M}$$

with $\bar{M} \in C_s([0,\rho];\Sigma^+(H))$.

Finally, if $\{P_{0k}\}$ *is an increasing sequence in* $\Sigma^+(H)$ *which converges strongly to* P_0 *then* $\wedge(P_{0k}, M)$ *converges to* $\wedge(P_0, M)$ *in* $C_s([0,\rho]\,;\Sigma^+(H))$.

The following equality is also easily proved (see [1]).

PROPOSITION 1.2. *Let* $P = \wedge(P_0, M)$ *with* $P_0 \in \Sigma^+(H)$, $M \in C_s([0,\rho];\Sigma^+(H))$. *Let* $u \in L^2(0,\rho\,;U)$ *and let* y *be the mild solution to the problem:*

(1.5)
$$y' = Ay + Bu\,, \ y(0) = x\,, \ x \in H$$

Then we have:

(1.6)
$$\langle P(t)\,x, x\rangle + \int_0^t \| B^*P(t-s)\,y(s) + u(s)\|^2\,ds = \int_0^t [\langle M(t-s)\,y(s), y(s)\rangle + \|u(s)\|^2]\,ds + \langle P_0\,y(t), y(t)\rangle$$

2. THE MAIN RESULT

Let us consider the Poincaré mapping γ:

(2.1)
$$\Sigma^+(H) \to \Sigma^+(H)\,, \ T \to \gamma(T) = \wedge(T, M)(\rho).$$

By definition a *periodic solution* of (1.1) of period ρ is a function $P \in C_s([0, \rho]; \Sigma^+(H))$ such that:

$$\gamma(P(0)) = P(0) \tag{2.2}$$

By (1.4) we have:

$$T \leq \overline{T}, T, \overline{T} \in \Sigma^+(H) \Rightarrow \gamma(T) \leq \gamma(\overline{T}). \tag{2.3}$$

Then setting

$$T_0 = 0, \; T_{n+1} = \gamma(T_n), \; n \in \mathbf{N} \tag{2.4}$$

$\{T_n\}$ is an increasing sequence in $\Sigma^+(H)$.

We shall assume now the following stabilizability condition:

$$\begin{cases} \text{There exists } K \in \mathscr{L}(H, U) \text{ such that} \\ \|e^{t(A-BK)}\| \leq N e^{-\omega t} \end{cases} \tag{2.5}$$

with $N > 0$ and $\omega > 0$.

THEOREM 2.1. *Assume* (1.2) *and* (2.5). *Then if* $N < e^{2\pi\omega}$ *there exsits a periodic solution of equation* (1.1).

Proof. Let $\{T_n\}$ be defined by (2.4). In virtue of the last affirmation of Proposition 1.1 and by a well-known result on the monotone sequences of linear bounded operators it suffices to show that $\{T_n\}$ is bounded. To this purpose we first remark that, by (1.6) we have:

$$\langle T_1 x, x \rangle \leq \int_0^\rho [\langle M(\rho - s) y(s), y(s) \rangle + \|u(s)\|^2] \, ds \tag{2.6}$$

and

$$\begin{aligned} \langle T_{n+1} x, x \rangle \leq \int_0^\rho [\langle M(\rho - s) y(s), y(s) \rangle + \|u(s)\|^2] \, ds + \\ + \langle T_n y(\rho) y(\rho) \rangle \; n \in \mathbf{N}. \end{aligned} \tag{2.7}$$

Set now in (2.6) and (2.7) $u(t) = -Ke^{(A-BK)t} x$, $y(t) = e^{(A-BK)t} x$; then by a recurrence argument we find:

$$\|T_{n+1}\| \leq 2\pi\mu \sum_{l=0}^{n} N^2 e^{-2\omega\rho} \leq 2\pi\tilde{\mu}/(1 - N^2 e^{-2\omega\rho}), \; \tilde{\mu} = \rho N^2 (\mu + \|K\|^2)$$

where $\mu = \sup_{t \in [0,\rho]} \|M(t)\|$. Thus the conclusion follows.

Remark 2.2. Theorem (2.1) generalize a result of Tartar [3].

Remark 2.3. Assume that $U = H$, $B = I$ and $\| e^{tA} \| \leq e^{\eta t}$, with $\eta \in \mathbf{R}$, then hypothesis (2.5) holds with $N = I$.

Remark 2.4. Similar results hold for the equation:

$$P'(t) = A^*(t) P(t) + P(t) A(t) - P(t) B^*(t) P(t) + M(t) \tag{2.8}$$

under the following hypotheses:

(2.9)

a) There exists a strongly continuous evolution operator $U_A(t, s)$ generated by the family $\{A(t)\}_{t \in [0, \rho]}$

b) $B \in C_s([0, \rho]; \mathscr{L}(U, H))$

c) $M \in C_s([0, \rho]; \mathscr{L}(H))$

d) There exists $K \in C_s([0, \rho]; \mathscr{L}(H, U))$ such that

$$\| U_{A-BK}(t, s) \| \leq N e^{-\omega(t-s)}, \quad N > 0, \quad \omega > 0$$

where U_{A-BK} is the evolution operator generated by the family $\{A(t) - B(t) K(t)\}_{t \in [0, \rho]}$

References

[1] V. Barbu and G. Da Prato (1983) – *Hamilton Jacobi equations in Hilbert spaces*, Pitman, London.

[2] G. Da Prato (1973) — *Quelques résultats d'existence, unicité and regularité pour un problème de la théorie du contrôle.* « J. Math. Pures et Appl. », *52*, 353-375.

[3] L. Tartar (1974) – *Sur l'étude direct d'équations non linéaires intérvenant en théorie du controle optimal*, « J. Functional Analysis », *18*, 1-47.