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A curve of genus q with a Half-Canonical embedding in $\mathbf{P}^3$

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Geometria. — *A curve of genus q with a Half–Canonical embedding in* $\mathbf{P}^3$. Nota (*) di SEVIN RECILLAS, presentata dal Socio G. ZAPPA.

RIASSUNTO. — Si costruiscono curve di genere $g = 4n - 3$, $n \geq 3$ che hanno $2^{n-3}(2^{n-2} - 1)$ fasci semicanonici L tali che $h^0(L) = 4$. Per $n + 3$ si dimostra che gli L sono molto ampi.

1. INTRODUCTION

It is classicaly known that a way to construct rank 2 vector bundles on $\mathbf{P}^3$ is to find curves $\tilde{\tilde{C}}$ which carry a very-ample half-canonical line bundle L such that $h^0(L) = 4$. Here we propose an example of a family (of dimension $3n - 3$, $n \geq 3$) of such curves of genus $g = 4n - 3$. The very-ampleness of L has only been proved in the case $g = q$.

2. CONSTRUCTION OF THE GENERAL EXAMPLE

Let C be a generic curve of genus n ($n \geq 3$) defined over the complex field $\mathbf{C}$. Let $\eta \in \mathrm{Pic}^0(C)$ be a non-zero element of order 2. One knows that there exists $2^{n-2}(2^{n-1} - 1)$ different pairs of odd theta-characteristics $\{L_1, L_2\}$ such that $L_1 \otimes L_2^{-1} = \eta$. Pick two such pairs $\{L_1, L_2\}$ and $\{L_3, L_4\}$. Let us observe that $L_1 \otimes L_3^{-1} = \sigma$ is also a point of order 2 on $\mathrm{Pic}^0(C)$ and $\{L_1, L_2, L_3, L_4\}$ is an homogeneous space for the group $\{0, \eta, \sigma, \eta\sigma\}$. We also recall that since C is general, one has $h^0(L_i) = 1$, $i = 1, 2, 3, 4$.

Let $\tilde{C} \xrightarrow{\alpha} C$ denote the unramified double cover associated to η, we will use the following remarks about such covers: If F is an invertible sheaf on C, then there is a natural isomorphism

[M] $$H^0(\tilde{C}, \alpha^* F) \cong H^0(C, F) \otimes H^0(C, F \otimes \eta).$$

If $\tilde{C}$ is hyperelliptic (elliptic-hyperelliptic) then the curve C is also hyperelliptic (elliptic-hyperelliptic) [D].

(*) Pervenuta all'Accademia il 26 settembre 1984.

Since our theta-characteristics differ by η, when we pull them back to $\tilde{C}$ we get $\alpha^* L_1 = \alpha^* L_2 = M$ and $\alpha^* L_3 = \alpha^* L_4 = N$ and since the cover is unramified they are still half-canonical on $\tilde{C}$ and in fact by the previous remark we know that

$$h^0(\tilde{C}, M) = h^0(\tilde{C}, N) = 2 .$$

Now $\alpha^* \sigma$ is a point of order 2 in $\mathrm{Pic}^0(\tilde{C})$ and $M = \alpha^* \sigma \otimes N$, so we can do the same construction again. Let $\tilde{\tilde{C}} \xrightarrow{\beta} \tilde{C}$ be the unramified double cover associated to $\alpha^* \sigma$, then $\beta^* M = \beta^* N = L$ is still half-canonical and we have $h^0(\tilde{\tilde{C}}, L) = 4$.

The couple $(\tilde{\tilde{C}}, L)$, is then a member of our family.

Let us observe that $\tilde{\tilde{C}}$ depends only on the subgroup $\{0, \eta, \sigma, \eta\sigma\} \subset \subset \mathrm{Pic}^0(C)$ and L depends only on the set $\{L_1, L_2, L_3, L_4\}$. From this and from the fact that on a generic C, for a given subgroup $\{0, \eta, \sigma, \eta\sigma\}$ there exist $2^{n-3}(2^{n-2} - 1)$ different sets (one computes this the same way as the number of odd theta pairs), it follows that on a given $\tilde{\tilde{C}}$ as above, there exist $2^{n-3}(2^{n-2} - 1)$ different half-canonical sheaves L such that $h^0(\tilde{\tilde{C}}, L) = 4$.

Another way of looking at such $\tilde{\tilde{C}}$ is as a general curve with a group of automorphisms isomorphic to $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ whose elements are fixed point free.

3. The Case $n = 3$

In this case $\tilde{C}$ is of genus 5 so $h(\tilde{C}, \Omega^{\frac{1}{2}}_{\tilde{C}} \alpha^* \sigma) = 4$, hence since $M \otimes N = \Omega^{\frac{1}{2}}_{\tilde{C}} \otimes \alpha^* \sigma$ the image of $\tilde{C}$ under the Prym-canonical map $\tilde{C} \to \mathbf{P}^3$ (which is birational since $\tilde{C}$ cannot be hyperelliptic or elliptic-hyperelliptic because we are assuming C general) is contained in a non-singular quadric surface whose rulings cut the linear systems $|M|$ and $|N.|$ Since $P_a(\tilde{C}) = 0$, this Prym-canonical curve must have some singularities, one knows that those can only be double points corresponding to divisors $a_i \in \tilde{C}^{(2)}$ $i = 1, 2, 3, 4$, which must be arranged in the form

$$a_1 \sim a_2 + \alpha^* \sigma \quad \text{and} \quad a_3 \sim a_4 + a^* \sigma \quad \text{with} \quad a_i \cap a_j = \Phi , \quad i \neq j .$$

Moreover, since the rulings of the quadric cut $|M|$ and $|N|$ and since $M = = N \otimes \alpha^* \sigma$ one must have

$$a_1 + a_3 , \ a_2 + a_4 \in |M| \qquad \text{and} \quad a_1 + a_4 , \ a_2 + a_3 \in |N| .$$

Also observe that if i denotes the involution on $\tilde{C}$, since $\alpha\, a_1 \sim \alpha\, a_2$ we must have (C being not hyperelliptic)

$$\alpha a_1 = \alpha a_2 \quad \text{i.e.} \quad a_1^i = a_2 \qquad \text{and also} \quad a_3^i = a_4 .$$

We lift now our divisors to $\tilde{\tilde{C}} \xrightarrow{\beta} \tilde{C}$ and we observe that $\beta^* a_1 \sim \beta^* a_2$, $\beta^* a_3 \sim \beta^* a_4$, $\beta^* a_1 \nsim \beta^* a_3$ and $\beta^* a_i \cap \beta^* a_j = \Phi$, $i \neq j$.

So on $\tilde{\tilde{C}}$ we have two different g_4^1's whose sum is $|L|$. From this it follows that the image of the map associated to $L : \tilde{\tilde{C}} \xrightarrow{\varphi} \mathbf{P}^3$ is contained on a non–singular quadric surface whose rulings pull back to the g_4^1's, so our map is birational since being of degree 2 would imply that $\tilde{\tilde{C}}$ is elliptic-hyperelliptic. Finally since $P_\alpha(\tilde{\tilde{C}}) = 9 = g(\tilde{\tilde{C}})$ if follows that φ is an isomorphism.

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