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Periodic solutions to a non-linear parametric differential equation of the third order

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Equazioni differenziali. — *Periodic solutions to a non–linear parametric differential equation of the third order.* Nota (*) di JAN ANDRES e JAN VORÁCEK, presentata dal Corrisp. R. CONTI.

RIASSUNTO. — Si dimostra un teorema di esistenza di soluzioni periodiche dell'equazione differenziale ordinaria del terzo ordine $x''' + a(t, x, x', x'')x'' + b(t, x, x', x'')x' + h(x) = e(t, x, x', x'')$ con le funzioni a, b e periodiche in t di periodo ω.

The equation to be discussed is

$$(1) \quad x''' + a(t, x, x', x'')x'' + b(t, x, x', x'')x' + h(x) = e(t, x, x', x'')$$

with continuous functions a b, h, e. Furthermore we assume the ω–periodicity in the variable t of a, b, e.

We proceed by the well-known Leray-Schauder fixed point technique [see e.g. [1]].

The equation (1) results for $p := 1$ from the following family of differential equations depending on the parameter p:

$$(2_p) \quad x''' + p[a(t, x, x', x'')x'' + b(t, x, x', x'')x' + (h(x) - cx)] + cx = \\ = \mathrm{pe}(t, x, x', x'')$$

with a suitable constant $c \neq 0$.

For the existence of ω–periodic solutions of (1) the following conditions are sufficient:

(i) All the ω–periodic solutions $x(t)$ of (2_p) and $x'(t)$, $x''(t)$ are for $0 \leq p \leq 1$ a priori bounded by a constant independent of p.

(*) Pervenuta all'Accademia il 21 settembre 1984.

(ii) The linear equation

$$x''' + cx = 0$$

[resulting from (2_p) for $p := 0$] has no ω–periodic solution different from identical zero.

It is clear that the last condition (ii) will be satisfied for any $c \neq 0$.

Putting

$$\omega_1 := \frac{\omega}{2\pi}$$

we have for every ω–periodic function $f(t)$ with square integrable f' and f'':

$$\int_t^{t+\omega} f'^2(s)\,ds \leq \omega_1^2 \int_t^{t+\omega} f''^2(s)\,ds \qquad \text{for all } t \tag{3}$$

[cf. e.g. [2]].

Substituting into $a(t, x, x', x'')$ for x a fixed solution $x(t)$ of (1), we obtain the composed function $a(t, x(t), x'(t), x''(t))$ of the variable t. This function we write simply $a_x(t)$; in the same sense we use the symbols $b_x(t)$, $h_x(t)$, $e_x(t)$.

LEMMA 1. *If the following inequalities*

$$| a(t, x, y, z) | \leq A, \tag{4}$$

$$| b(t, x, y, z) | \leq B. \tag{5}$$

$$| e(t, x, y, z) | \leq E \tag{6}$$

hold for all t, x, y, z, *then for any*

$$\theta := 1 - \omega_1 (A + \omega_1 B) > 0 \tag{7}$$

every ω–periodic solution $x(t)$ *of* (2_p) *satisfies the inequality*

$$\int_t^{t+\omega} x''^2(s)\,ds \leq \omega \left[\frac{E\omega_1}{\theta}\right]^2 := D_2^2 \tag{8}$$

for all t.

Proof. Substituting a fixed $x(t)$ into (2_p), multiplying the obtained identity by $x'(t)$ and integrating we come to

$$\int_t^{t+\omega} x''^2(s)\,\mathrm{d}s = p \int_t^{t+\omega} [a_x(s)\,x''(s)\,x'(s) + b_x(s)\,x'^2(s) - e_x(s)\,x'(s)]\,\mathrm{d}s$$

and hence by (4), (5), (6)

$$\int_t^{t+\omega} x''^2(s)\,\mathrm{d}s \leq \int_t^{t+\omega} [\mathrm{A}\mid x''(s)\,x'(s)\mid + \mathrm{B}x'^2(s) + \mathrm{E}\mid x'(s)\mid]\,\mathrm{d}s\,.$$

Thereof by (3) and the Schwarz inequality it results

$$\theta^2 \int_t^{t+\omega} x''^2(s)\,\mathrm{d}s \leq (\mathrm{E}\,\omega_1)^2\,\omega$$

for all t, i.e. (8).

COROLLARY. *If all the assumptions of Lemma* 1 *are fulfilled, we get from* (8) *and* (3) *also*

$$\int_t^{t+\omega} x'^2(s)\,\mathrm{d}s \leq \omega_1^2\,\mathrm{D}_2^2 := \mathrm{D}_1^2 \qquad \textit{for all } t. \tag{9}$$

Since for some $t \leq t_1 \leq t + \omega$ *it must be*

$$x'(t_1) = 0$$

and consequently

$$x'(s) = \int_{t_1}^{s} x''(u)\,\mathrm{d}u\,,$$

we obtain from the Schwarz inequality that

$$\mid x'(s)\mid \leq \sqrt{\omega}\,\mathrm{D}_2 := \mathrm{D}' \tag{10}$$

holds for every real s.

LEMMA 2. *If all the assumptions of Lemma* 1 *are fulfilled and if moreover there exist such real numbers* $c \neq 0$, $h > 0$ *that*

$$xh(x)\,\mathrm{sgn}\,c \leq \mid c\mid x^2 \qquad \textit{for every } \mid x\mid \geq h \tag{11}$$

holds, then every ω*–periodic solution* $x(t)$ *of* (2_p) *satisfies*

$$| x(t) | < R + | D' | \omega := D \tag{12}$$

with

$$R := \max \left[h , \frac{A | D_2 | + B | D_1 |}{c \sqrt{\omega}} + \frac{E}{c} \right] \tag{13}$$

and henceforth also

$$\int_t^{t+\omega} x^2(s)\, ds < \omega D^2 := D_0^2 , \qquad \textit{for every } t . \tag{14}$$

Proof. We substitute again $x(t)$ into (2_p). Multiplying the resulting identity by $x(t)$ and integrating, we obtain for every t:

$$p \int_t^{t+\omega} [a_x(s)\, x''(s) + b_x(s)\, x'(s) - e_x(s)]\, x(s)\, ds = (p-1) \int_t^{t+\omega} cx^2(s)\, ds - \\ - p \int_t^{t+\omega} x(s)\, h_x(s)\, ds$$

and further [cf. (11) and (4)-(6)]

$$\int_t^{t+\omega} [(1-p)\, | c |\, x^2(s) + px(s)\, h_x(s) \operatorname{sgn} c]\, ds \leq \\ \leq \int_t^{t+\omega} [A\, | x''(s)\, x(s) | + B\, | x'(s)\, x(s) | + E\, | x(s) |]\, ds . \tag{15}$$

If on $\langle t , t + \omega \rangle$ the inequality $| x(s) | > R$ holds, we have

$$c^2 R^2 \omega < c^2 \int_t^{t+\omega} x^2(s)\, ds \leq (A\, | D_2 | + B\, | D_1 | + E \sqrt{\omega})^2$$

and this contradicts to (13). Thus on $\langle t , t + \omega \rangle$ there must exist a point t_1 with $| x(t_1) | \leq R$ and the lemma is proved.

LEMMA 3. *If all the assumptions of the foregoing lemma are valid, then denoting*

$$H := \max_{|x| \leq D} | h(x) |$$

we have for an arbitrary ω*-periodic solution* $x(t)$ *of* (2_p)

$$\int_t^{t+\omega} x'''^2(s)\, ds \leq (A \mid D_2 \mid + B \mid D_1 \mid + (H + E) \sqrt{\omega})^2 := D_3^2 \tag{16}$$

and thus [cf. (10)]

$$| x''(t) | \leq \sqrt{\omega}\, D_3 := D'' \tag{17}$$

for every t.

The proof of this Lemma may be performed by integration of the identity resulting from (2_p) multiplied by $x'''(t)$ similarly as in Lemma 1 and Lemma 2.

THEOREM. *If the conditions* (4)-(7) *are satisfied and if there exist such reals* $c \neq 0$, $h > 0$ *that* (11) *holds, then the equation* (1) *admits an* ω*-periodic solution.*

Proof. By Lemmas 1, 2, 3 we have for ω-periodic solution $x(t)$ of (2_p)

$$| x(t) | + | x'(t) | + | x''(t) | \leq D + | D' | + | D'' | := P$$

with P independent on $p \in \langle 0, 1 \rangle$.

Thus both conditions (i), (ii) sufficient for the existence of an ω-periodic solution of (1) are fulfilled.

Remark. Simplifying (1) into the equation

$$x''' + [a(t)\, x']' + g(x)\, x' + h(x) = e(t) \tag{18}$$

with ω-periodic $a(t)$ and $e(t)$, it is possible to require instead of (11) other conditions less restrictive with respect to $h(x)$.

The equation (18) has an ω-periodic solution, if besides (4), (5), (7) the following relations are satisfied:

$$h(x) \operatorname{sgn} x \gtrless 0 \qquad \text{for every} \quad | x | \geq h, \tag{19}$$

$$\int_0^{\omega} e(t)\, dt = 0\,. \tag{20}$$

Indeed, assuming $\operatorname{sgn} x(t) = \text{const.}$ on $\langle t, t+\omega\rangle$, we obtain from (18_p) [cf. (2_p)] and (20):

$$\int_t^{t+\omega} [h(x(s)) \operatorname{sgn} x(s) + (1-p)\, c \,|\, x(s) \,|]\, \mathrm{d}s = 0$$

for any c; nevertheless (19) implies for $|\, x(t) \,| \geq h$ on $\langle t, t+\omega\rangle$ and a suitable $c \neq 0$:

$$\int_t^{t+\omega} [h(x(s)) \operatorname{sgn} x(s) + (1-p)\, c \,|\, x(s) \,|]\, \mathrm{d}s \neq 0\,.$$

Hence there must be a point $t_1 \in \langle t, t+\omega\rangle$ such that $|\, x(t_1) \,| < h$.

References

[1] R. Reissig (1971) – *Ljapunovsche Funktionen und periodische Lösungen in nichtautonomen Systemen*, « Ist. Naz. Alta Mat. (Bologna) Symp. Mat. », (6), 361-395.
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