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A discretization method for the problem of a membrane constrained by elastic obstacle

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Meccanica dei solidi. — *A discretization method for the problem of a membrane constrained by elastic obstacles* (*). Nota (**) di ALDO MACERI (***), presentata dal Corrisp. E. GIANGRECO.

RIASSUNTO. — In questo lavoro sono dati alcuni modelli matematici per il problema di contatto tra una membrana ed un suolo od ostacolo elastico. Viene costruita una approssimazione lineare a tratti della soluzione e, tramite una disequazione variazionale discreta, se ne dà il corrispondente teorema di convergenza.

1. Many contact mathematical problems between elastic and non-elastic bodies were recently analyzed. Such models of some physical systems frequently encountered by engineers were deeply examined in connection with the growing development of the convex analysis and of the variational inequalities theory.

In particular, some membrane contact problems, involving second order differential operators, have been widely discussed. The case of a membrane constrained or forced by a rigid obstacle is a classical one [1]. The contact between two membranes transversely loaded was also examined, both from the point of view of existence, uniqueness and regularity of the solution [2], and of its numerical analysis [3].

The problem posed by an " elastic " (in some elementary sense) obstacle bounding the displacements of a membrane was analyzed in [4], [5] as far as existence, uniqueness and regularity of solutions were concerned, and in [6], to give a computational procedure for simple cases.

This paper is devoted to the analysis of the contact problem of a membrane stretched by an elastic obstacle (in particular, constrained by a subgrade, elastic in the Winkler's sense), which appears to be still not completely studied. We give first some mathematical models of the continuous problem, and we discuss their equivalence. A finite element discretization procedure of a variational non-linear model is then described, and we prove its convergence.

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2. Let us consider a plane membrane, and let us denote by Ω the region of R^2 occupied by the membrane. We assume that Ω is open, bounded and of class $\mathscr{R}^{(1),1}$ [7]. The membrane is fixed at the boundary points of Ω, is transversely loaded by distributed forces $f \in L^2(\Omega)$, orthogonal to its plane and positive in the x_3–axis direction, and is uniformly stretched in its plane by a stress $t \in]0, +\infty[$.

Let us denote as $u(x_1, x_2)$ the membrane's deflection, positive in the x_3–axis direction, and let us assume u to be "small" in the usual sense. Furthermore, the membrane is stretched (or constrained) by an elastic body. We describe the body shape by a function $\varphi \in L^2(\Omega)$, and we assume its reactions on the membrane to be parallel to the x_3-axis (frictionless contact) and to have the form (1) $-h(u-\varphi)^+$, where $h \in L^\infty(\Omega)$, $h \geq 0$ a.e. on Ω. In particular, $\varphi = 0$ corresponds to the case of a Winkler's subgrade.

The problem is to find the membrane's equilibrium configuration, i.e., let

$$A = -t\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)$$

Problem 1

$$u \in H_0^1(\Omega) \cap H^2(\Omega): \quad Au + h(u-\varphi)^+ = f \qquad \text{a.e.} \quad \text{on } \Omega.$$

This contact problem can also be formulated in a different way. In fact. it is easy to show that the complementarity problem

Problem 2

$$u \in H_0^1(\Omega) \cap H^2(\Omega): \begin{cases} Au - f \leq 0 & \text{a.e. on } \Omega \\ Au + h(u-\varphi) - f \leq 0 & \text{a.e. on } \Omega \\ (Au - f)(Au + h(u-\varphi) - f) = 0 & \text{a.e. on } \Omega \end{cases}$$

is satisfied by each solution of Problem 1) and vice versa.

In [4], the proof that Problem 2) has a unique solution u is given. Moreover, a regularity Theorem [5] ensures that $u \in H^{2,p}(\Omega)$ if $f, \varphi \in L^p(\Omega)$, $p > 2$, and that $u \in H^{3,p}(\Omega)$ under stronger hypotheses on f, φ, h, Ω.

But, both previous statements of the mechanical problem are nearly useless for the purpose of numerical computations. To this aim, the use of variational formulations appears to be more adequate. Then, let us set

$$a(u, v) = t \int_\Omega \left(\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\right) dx \qquad \forall (u, v) \in (H_0^1(\Omega))^2$$

(1) We let $v^+ = \max\{v, 0\}$.

$$\langle F , v \rangle = \int_\Omega f v \, dx \qquad \forall \, v \in L^2(\Omega)$$

$$E_2(v) = \frac{1}{2} \int_\Omega h \, [(v - \varphi)^+]^2 \, dx \qquad \forall \, v \in L^2(\Omega) .$$

Obviously, $\exists \, c_1 \in \,]0 , +\infty[$ such that

(1) $$a(v , v) \geq c_1 \| v \|^2_{H^1(\Omega)} \qquad \forall \, v \in H^1_0(\Omega) .$$

Moreover [8], the functional E_2 is convex and Gateaux differentiable in $H^1_0(\Omega)$.

Let us consider now the total energy minimum problem

Problem 3

$$u \in H^1_0(\Omega) : \frac{1}{2} a(u , u) - \langle F , u \rangle + E_2(u) \leq \frac{1}{2} a(v , v) - \langle F , v \rangle + \\ + E_2(v) \qquad \forall \, v \in H^1_0(\Omega)$$

and the (virtual work) variational equation

Problem 4

$$u \in H^1_0(\Omega) : a(u , v) + \int_\Omega h (u - \varphi)^+ v \, d\,x - \langle F , v \rangle = 0 \qquad \forall \, v \in H^1_0(\Omega)$$

and the mixed type variational inequality

Problem 5

$$u \in H^1_0(\Omega) : a(u , v - u) + E_2(v) - E_2(u) - \langle F , v - u \rangle \geq 0 \qquad \forall \, v \in H^1_0(\Omega).$$

To prove that these problems are variational formulations of Problem 1, we need a regularity result. Therefore, for every $\delta > 0$ we let $S_\delta = \{ y \in R^2 : | y | < \delta \}$, $\Sigma_\delta = \{ (y_1 , y_2) \in S_\delta : y_2 > 0 \}$, $H(\delta) = \{ v \in L^2(\Sigma_\delta) : \exists \, \delta_v \in \,]0 , \delta[$ such that $(| y | > \delta_v) \Rightarrow (v(y) = 0)$ *a. e.* on $\Sigma_\delta \}$ and we notice with s_δ the curvilinear measure on $\partial \Sigma_\delta$ [7]. Furthermore, we consider the bilinear integrodifferential form $b(u , v) = \sum\limits_{\substack{|r| \leq 1 \\ |s| \leq 1}} \int_{\Sigma_\delta} b_{rs} D^s u \, D^r v \, dy \quad \forall \, (u , v) \in (H^1(\Sigma_\delta))^2$ where $b_{rs} \in L^\infty(\Sigma_\delta)$. By proceeding as in [8], we prove

LEMMA 1. *Let us suppose*

$$b_{rs} \in C^{0,1}(\overline{\Sigma_\delta}) \,,\ u \in H^1(\Sigma_\delta) \,,\ u = 0 \quad s_\delta\text{—a.e.} \quad \text{on} \quad \{(y_1, y_2) \in \Sigma_\delta : y_2 = 0\} \,,$$

$$\exists c_2 \in]0, +\infty[\, : b(v, v) \geq c_2 \int_{\Sigma_\delta} \left(\left(\frac{\partial v}{\partial y_1} \right)^2 + \left(\frac{\partial v}{\partial y_2} \right)^2 \right) dy \quad \forall v \in H_0^1(\Sigma_\delta) \cap H(\delta) \,,$$

$$\exists c_3 \in]0, +\infty[\, : | b(u, v) | \leq c_3 \| v \|_{L^2(\Sigma_\delta)} \quad \forall v \in H_0^1(\Sigma_\delta) \cap H(\delta) \,.$$

Then

$$(\delta' \in]0, \delta[) \Rightarrow (u \in H^2(\Sigma_{\delta'})) \,.$$

Now we can prove

THEOREM 1. *The following statements are equivalent*

u is solution of Problem 1;

u is solution of Problem 3;

u is solution of Problem 4;

u is solution of Problem 5.

Proof. It is sufficient [9] to prove that every solution of Problem 4) belongs to $H^2(\Omega)$.

Let $z \in \partial\Omega$. Then [7] an open neighbourhood Z of z, a $\delta \in]0, +\infty[$ and an invertible application $\Psi = (\Psi_1, \Psi_2)$ of S_δ onto Z exist such that $\Psi \in C^{1,1}(S_\delta)$, $\Psi^{-1} \in C^{1,1}(Z)$, $\Psi(\Sigma_\delta) = Z^+$ were $Z^+ = \Omega \cap Z$, $\left| \dfrac{\partial(\Psi_1, \Psi_2)}{\partial(y_1, y_2)}(y) \right| = 1 \ \forall \ y \in S_\delta$.

For every $(\tilde{v}, \tilde{w}) \in (H_0^1(\Sigma_\delta))^2$, if we let $v = \tilde{v} \circ \Psi^{-1}$ and $w = \tilde{w} \circ \Psi^{-1}$, we have [9]

$$t \int_{Z^+} \left(\frac{\partial v}{\partial x_1} \frac{\partial w}{\partial x_1} + \frac{\partial v}{\partial x_2} \frac{\partial w}{\partial x_2} \right) dx = \sum_{\substack{|\alpha| \leq 1 \\ |\beta| \leq 1}} \int_{\Sigma\delta} b_{\alpha\beta}(y) \, D^\beta \tilde{v}(y) \, D^\alpha \tilde{w}(y) \, dy = b(\tilde{v}, \tilde{w})$$

where $b_{\alpha\beta} \in C^{0,1}(\Sigma_\delta)$ is independent from $(\tilde{v}, \tilde{w})$.

Now we observe that, if $\tilde{v} \in H_0^1(\Sigma_\delta) \cap H(\delta)$ and we let $v = \tilde{v} \circ \Psi^{-1}$ on Z^+ and $v = 0$ on $\Omega - Z^+$, because $v \in H_0^1(\Omega)$ and

$$\sum_{|r|=1} \int_{\Sigma\delta} | D^r \tilde{v} |^2 \, dy \leq c_4 \sum_{|r|=1} \int_{Z^+} | D^r v |^2 \, dx \,,$$

where $c_4 \in]0, +\infty[$ does not depend on $\tilde{v}$, we have

$$b(\tilde{v}, \tilde{v}) = \sum_{\substack{|r|=1\\|s|=1}} t \int_{Z^+} D^s v(x) D^r v(x) \, dx = a(v, v) \geq c_1 \sum_{|r|=1} \int_{Z^+} | D^r v |^2 dx \geq$$

$$\geq \frac{c_1}{c_4} \sum_{|\alpha|=1} \int_{\Sigma\delta} | D^\alpha \tilde{v} |^2 dy .$$

Moreover $\forall \tilde{v} \in H_0^1(\Sigma_\delta) \cap H(\delta)$, by putting $\tilde{u} = u \circ \Psi$, $v = \tilde{v} \circ \Psi^{-1}$ on Z^+, $v = 0$ on $\Omega - Z^+$, because $a(u, v) = \langle F, v \rangle - \int_\Omega f(u - \varphi)^+ v \, dx$, we have

$$| b(\tilde{u}, \tilde{v}) | \leq (\| f \|_{L^2(\Omega)} + \| h \|_{L^\infty(\Omega)} \| (u - \varphi)^+ \|_{L^2(\Omega)}) \| v \|_{L^2(Z^+)} \leq c_5 \| \tilde{v} \|_{L^2(\Sigma_\delta)}$$

where $c_5 \in]0, +\infty[$ does not depend on $\tilde{v}$.

As a consequence, by using Lemma 1, $\forall \delta' \in]0, \delta[$ $\tilde{u} \in H^2(\Sigma_{\delta'})$. Regularity on the boundary then follows. Interior regularity being known [10], we have $u \in H^2(\Omega)$.

The last three equivalent formulations appear to be very useful to generate discrete models of the membrane contact problem, mainly because of their need of a very simple functional space, $H_0^1(\Omega)$. This will be made in the next section, with a particular reference to the approximation of Problem 5.

3. Let $n \in N$. Let $\mathcal{T}_n$ be a finite family of closed triangles of R^2 such that, $\forall T \in \mathcal{T}_n$, $T \subseteq \bar{\Omega}$ and meas $(T) > 0$, and such that $\forall T_1, T_2 \in \mathcal{T}_n$ $T_1 \cap T_2$ is empty or is $\{a\}$ where a is a vertex of T_1 and of T_2 or is an edge of T_1 and of T_2. Moreover, we let $\Omega_n = \bigcup_{T \in \mathcal{T}_n} T$, $l_n = \sup_{T \in \mathcal{T}_n} \operatorname{diam}(T)$, $s_n = \sup_{T \in \mathcal{T}_n} \operatorname{diam}(T) / \sup \{\operatorname{diam}(C) : C$ closed circle $\subseteq T\}$, $I_n = \{x \in \bar{\Omega} : x$ is a vertex of a $T \in \mathcal{T}_n\}$, $I'_n = \{x \in I_n : x \notin \partial\Omega_n\}$,

$$\mathcal{F}'_n = \{T \in \mathcal{T}_n : \text{a vertex of } T \in \partial\Omega_n\}, \quad \Omega'_n = \bigcup_{T \in \mathcal{F}'_n} T$$

and suppose

$$\lim_{n \to +\infty} l_n = 0 ; \tag{2}$$

$\exists c_6 \in]0, +\infty[: \forall m \in N$ $s_m \leq c_6$; $\forall$ W compact $\subseteq \Omega$ $\exists \nu \in N : W \subseteq \Omega_m$ $\forall m > \nu$.

Last two hypotheses imply

$$\lim_{n \to +\infty} \operatorname{meas}(\Omega - \Omega_n) = 0 , \lim_{n \to +\infty} \operatorname{meas}(\Omega'_n) = 0. \tag{3}$$

Let us now introduce the space P_1 given by not greater than 1st degree polynomials of R^2 and let us denote, with $a_1, \ldots, a_{m_n}$ the elements of I'_n. Moreover, we denote $\forall i \in \{1, \ldots, m_n\}$ with g_{ni} the element of $C^0(\bar{\Omega})$

such that $g_{ni}(a_i) = 1$, $g_{ni}(a) = 0$ $\forall a \in I_n - \{a_i\}$, $g_{ni}|_T \in P_1$ $\forall T \in \mathcal{T}_n$ and we introduce the subspace H_n of $H^1_0(\Omega)$

$$\left\{ \sum_{i=1}^{m_n} X_i g_{ni} : X_i \in R \right\}.$$

After that, $\forall v \in C^0(\overline{\Omega})$, let us denote with $r_n v$ the element of H_n such that $r_n v(x) = v(x)$ $\forall x \in I'_n$. Moreover, we let

$$\varepsilon_n = \max \{l_n^{1/4}, \text{meas}^{1/4}(\Omega - \Omega_n), \text{meas}^{1/4}(\Omega'_n)\}, \tag{4}$$

$$\varphi_n = r_n J_{\varepsilon_n} * \varphi$$

and recall [7] that $J_{\varepsilon_n} * \varphi \in C^\infty(R^2)$ and

$$\lim_{n \to +\infty} \| J_{\varepsilon_n} * \varphi - \varphi \|_{L^2(\Omega)} = 0. \tag{5}$$

Finally, we let

$$\varphi_n = \sum_{i=1}^{m_n} \Phi_{ni} g_{ni},$$

$$E_{2n}(v_n) = \frac{1}{2} \int_\Omega h \left[\sum_{i=1}^{m_n} (V_{ni} - \Phi_{ni})^+ g_{ni}(x) \right]^2 dx \quad \forall v_n = \sum_{i=1}^{m_n} V_{ni} g_{ni} \in H_n$$

and consider the mixed type variational inequality

Problem 5'

$$u_n \in H_n : a(u_n, v_n - u_n) + E_{2n}(v_n) - E_{2n}(u_n) - \langle F, v_n - u_n \rangle \geq 0 \quad \forall v_n \in H_n.$$

We have

THEOREM 2. *Problem 5' allows a unique solution.*

Proof. Uniqueness is obvious. About the existence, because

$$\lim_{\substack{v \in H_n \\ \|v\|_{H^1(\Omega)} \to +\infty}} \left\{ a(v, v) + \frac{1}{2} \int_\Omega h \left[\sum_{i=1}^{m_n} (V_{ni} - \Phi_{ni})^+ g_{ni} \right]^2 dx - \frac{1}{2} \int_\Omega h \left[\sum_{i=1}^{m_n} (-\Phi_{ni})^+ g_{ni} \right]^2 dx \right\} \| v \|^{-1}_{H^1(\Omega)} = +\infty,$$

it is sufficient [11] to prove that

$$E_{2n} - \frac{1}{2} \int_\Omega h \left[\sum_{i=1}^{m_n} (-\Phi_{ni})^+ g_{ni} \right]^2 dx \tag{6}$$

is weakly lower-semicontinuous. Because H_n has finite dimension, for every convergent sequence $\left\{\sum_{i=1}^{m_n} V_{jni}\, g_{ni}\right\}$, whose limit we notice with $\sum_{i=1}^{m_n} V_{ni}\, g_{ni}$, we have

$$\lim_{j\to+\infty}\left\| h^{1/2}\left[\sum_{i=1}^{m_n}(V_{jni}-\Phi_{ni})^+ g_{ni}-\sum_{i=1}^{m_n}(V_{ni}-\Phi_{ni})^+ g_{ni}\right]\right\|_{L^2(\Omega)}=0\,.$$

As a consequence,

$$\forall\mu\in R\quad\left\{v\in H_n : E_{2n}(v)-\frac{1}{2}\int_{\Omega} h\left[\sum_{i=1}^{m_n}(-\Phi_{ni})^+ g_{ni}\right]^2 dx\leq\mu\right\}$$

is closed in $H^1(\Omega)$ and this result implies (6).

Let us notice with $u_n=\sum_{i=1}^{m_n} U_{ni}\, g_{ni}$ the solution of Problem 5′. Before proving that $\{u_n\}$ is convergent towards the true solution u of Problem 1, we establish some Lemmas.

LEMMA 2. *It results*

i) $$\lim_{n\to+\infty}\int_{\Omega-\Omega_n}(J_{\varepsilon_n}*\varphi)^2\,dx=0$$

ii) $\forall n\in N$ and $\forall i\in\{1\ ,\ 2\}$ $\max_{x\in\bar\Omega}\left|\frac{\partial I_{\varepsilon_n}*\varphi}{\partial x_i}(x)\right|\leq l_n^{-1/2}\,\|\varphi\|_{L^2(\Omega)}\left\|\frac{\partial J}{\partial x_i}\right\|_{L^2(R^2)}$.

Proof. Let $n\in N$. We have

$$\forall x\in\bar\Omega\quad |J_{\varepsilon_n}*\varphi(x)|=\left|\int_{R^2}\varepsilon_n^{-2}\,J\left(\frac{x-y}{\varepsilon_n}\right)\varphi(y)\,dy\right|\leq$$

$$\leq\varepsilon_n^{-2}\,\|\varphi\|_{L^2(\Omega)}\left(\int_{R^2}J^2\left(\frac{x-y}{\varepsilon_n}\right)dy\right)^{1/2}=\varepsilon_n^{-1}\,\|\varphi\|_{L^2(\Omega)}\,\|J\|_{L^2(R^2)}$$

from which, taking account of (4),

$$\int_{\Omega-\Omega_n}(J_{\varepsilon_n}*\varphi)^2\,dx\leq\varepsilon_n^{-2}\,\|\varphi\|^2_{L^2(\Omega)}\,\|J\|^2_{L^2(R^2)}\ \text{meas}\ (\Omega-\Omega_n)\leq$$

$$\leq\|\varphi\|^2_{L^2(\Omega)}\,\|J\|^2_{L^2(R^2)}\ \text{meas}^{1/2}(\Omega-\Omega_n)\,;$$

this result, because of (3), implies i).

About ii), let $n \in \mathrm{N}$ and $i \in \{1, 2\}$. We have, $\forall x \in \overline{\Omega}$,

$$\left|\frac{\partial \mathrm{J}_{\varepsilon_n} * \varphi}{\partial x_i}(x)\right| = \left|\int_{\mathrm{R}^2} \varphi(y) \frac{\partial \mathrm{J}_{\varepsilon_n}}{\partial x_i}(x-y)\, \mathrm{d}y\right| \le \varepsilon_n^{-2} \| \varphi \|_{\mathrm{L}^2(\Omega)} \left\| \frac{\partial \mathrm{J}}{\partial x_i} \right\|_{\mathrm{L}^2(\mathrm{R}^2)}$$

As a consequence, because of (4), the thesis follows.

Now, $\forall n \in \mathrm{N}$ and $\forall \mathrm{T} \in \mathscr{T}_n$ let us notice with a, b, c the vertices of T and, $\forall x \in \mathrm{T}$, with $t_a(x)$, $t_b(x)$, $t_c(x)$ the barycentric coordinates of x, so that [12]

$$t_a|_{\mathrm{T}}, t_b|_{\mathrm{T}}, t_c|_{\mathrm{T}} \in \mathrm{P}_1 \tag{7}$$

$$t_a(a) = t_b(b) = t_c(c) = 1, \; t_a(b) = t_a(c) = t_b(a) = t_b(c) = t_c(a) = t_c(b) = 0 \tag{8}$$

$$t_a(x) + t_b(x) + t_c(x) = 1 \qquad \forall x \in \mathrm{T} \tag{9}$$

$$\forall p \in \mathrm{P}_1 \qquad p(x) = p(a)\, t_a(x) + p(b)\, t_b(x) + p(c)\, t_c(x) \quad \forall x \in \mathrm{T}. \tag{10}$$

We have

LEMMA 3. *It results*

$$\lim_{n\to+\infty} \| \varphi_n - \mathrm{J}_{\varepsilon_n} * \varphi \|_{\mathrm{L}^2(\Omega)} = 0 .$$

Proof. At first we notice that $\forall n \in \mathrm{N}$, $\forall \mathrm{T} \in \mathscr{T}_n$ and $\forall x \in \mathrm{T}$ it results (2)

$$| \mathrm{J}_{\varepsilon_n} * \varphi(a) - \mathrm{J}_{\varepsilon_n} * \varphi(x) | + | \mathrm{J}_{\varepsilon_n} * \varphi(b) - \mathrm{J}_{\varepsilon_n} * \varphi(x) | + | \mathrm{J}_{\varepsilon_n} * \varphi(c) - \mathrm{J}_{\varepsilon_n} * \varphi(x) | \le 3\, l_n \left(\max_{x\in\overline{\Omega}} \left| \frac{\partial \mathrm{J}_{\varepsilon_n} * \varphi}{\partial x_1}(x) \right| + \max_{x\in\overline{\Omega}} \left| \frac{\partial \mathrm{J}_{\varepsilon_n} * \varphi}{\partial x_2}(x) \right| \right).$$

As a consequence, taking account of (7), (8), (9), (10) and Lemma 2,

$$\int_{\Omega} | \varphi_n - \mathrm{J}_{\varepsilon_n} * \varphi |^2\, \mathrm{d}x = \int_{\Omega_n} | r_n \mathrm{J}_{\varepsilon_n} * \varphi(x) - \mathrm{J}_{\varepsilon_n} * \varphi(x) |^2\, \mathrm{d}x + \int_{\Omega-\Omega_n} (\mathrm{J}_{\varepsilon_n} * \varphi)^2\, \mathrm{d}x \le$$

$$\le 2 \sum_{\mathrm{T}\in\mathscr{T}_n} \int_{\mathrm{T}} | \mathrm{J}_{\varepsilon_n} * \varphi(a)\, t_a(x) + \mathrm{J}_{\varepsilon_n} * \varphi(b)\, t_b(x) + \mathrm{J}_{\varepsilon_n} * \varphi(c)\, t_c(x) -$$

(2) Clearly $\forall n \in \mathrm{N}$, $\forall \mathrm{T} \in \mathscr{T}_n$ and $\forall g \in \mathrm{C}^\infty(\overline{\Omega})$, $\forall x \in \mathrm{T}$ $| g(a) - g(x) | + | g(b) - g(x) | + | g(c) - g(x) | \le 3 l_n \left(\max_{x\in\overline{\Omega}} \left| \frac{\partial g}{\partial x_1}(x) \right| + \max_{x\in\overline{\Omega}} \left| \frac{\partial g}{\partial x_2}(x) \right| \right)$.

$$-\mathrm{J}_{\varepsilon_n} * \varphi(x)\left(t_a(x)+t_b(x)+t_c(x)\right)|^2\, dx+\int_{\Omega-\Omega_n}(\mathrm{J}_{\varepsilon_n} * \varphi)^2\, dx+$$

$$+18 \text{ meas}(\Omega'_n) \max_{x\in\overline{\Omega}}{}^2 |\mathrm{J}_{\varepsilon_n} * \varphi(x)| \leq$$

$$\leq 36\, l_n \|\varphi\|^2_{L^2(\Omega)}\left(\left\|\frac{\partial \mathrm{J}}{\partial x_1}\right\|^2_{L^2(R^2)}+\left\|\frac{\partial \mathrm{J}}{\partial x_2}\right\|^2_{L^2(R^2)}\right) \text{meas}(\Omega)+\int_{\Omega-\Omega_n}(\mathrm{J}_{\varepsilon_n} * \varphi)^2\, dx+$$

$$+18 \text{ meas}^{1/2}(\Omega'_n)\|\varphi\|^2_{L^2(\Omega)}\|\mathrm{J}\|^2_{L^2(R^2)}$$

and thus the thesis follows because of (2), (3) and Lemma 2.

LEMMA 4. *For every* $w \in C_0^\infty(\Omega)$ *we have*

$$\lim_{n\to+\infty} \mathrm{E}_{2n}(r_n w)=\mathrm{E}_2(w).$$

Proof. Let us notice that (5) implies [7]

$$\lim_{n\to+\infty}\|h^{1/2}(w-\mathrm{J}_{\varepsilon_n} * \varphi)^+-h^{1/2}(w-\varphi)^+\|_{L^2(\Omega)}=0. \tag{11}$$

On the other hand, $\forall n\in \mathrm{N}$, $\forall \mathrm{T}\in\mathscr{T}_n$ and $\forall x\in \mathrm{T}$ we have [2] [3]

$$|[w(a)-\mathrm{J}_{\varepsilon_n} * \varphi(a)]^+ t_a(x)+[w(b)-\mathrm{J}_{\varepsilon_n} * \varphi(g)]^+ t_b(x)+[w(c)-$$

$$-\mathrm{J}_{\varepsilon_n} * \varphi(c)]^+ t_c(x)-[w(x)-\mathrm{J}_{\varepsilon_n} * \varphi(x)]^+| \leq |[w(a)-\mathrm{J}_{\varepsilon_n} * \varphi(a)]^+-$$

$$-[w(x)-\mathrm{J}_{\varepsilon_n} * \varphi(x)]^+|+|[w(b)-\mathrm{J}_{\varepsilon_n} * \varphi(b)]^+-[w(x)-\mathrm{J}_{\varepsilon_n} * \varphi(x)]^+|+$$

$$+|[w(c)-\mathrm{J}_{\varepsilon_n} * \varphi(c)]^+-[w(x)-\mathrm{J}_{\varepsilon_n} * \varphi(x)]^+| \leq 3\, l_n\left(\max_{x\in\overline{\Omega}}\left|\frac{\partial w}{\partial x_1}(x)\right|+\right.$$

$$\left.+\max_{x\in\overline{\Omega}}\left|\frac{\partial w}{\partial x_2}(x)\right|+\max_{x\in\overline{\Omega}}\left|\frac{\partial \mathrm{J}_{\varepsilon_n} * \varphi}{\partial x_1}(x)\right|+\max_{x\in\overline{\Omega}}\left|\frac{\partial \mathrm{J}_{\varepsilon_n} * \varphi}{\partial x_2}(x)\right|\right).$$

From this result, because of Lemma 2, it follows

$$\forall n\in \mathrm{N}\quad \int_\Omega\left|h^{1/2}\left[\sum_{i=1}^{m_n}(\mathrm{W}_{ni}-\Phi_{ni})^+ g_{ni}\right]-h^{1/2}(w-\mathrm{J}_{\varepsilon_n} * \varphi)^+\right|^2 dx\leq$$

$$\leq 2\|h\|_{L^\infty(\Omega)}\sum_{\mathrm{T}\,\mathscr{T}_n}\int_{\mathrm{T}}\left|\left[w(a)-\mathrm{J}_{\varepsilon_n} * \varphi(a)\right]^+ t_a(x)+\left[w(b)-\mathrm{J}_{\varepsilon_n} * \varphi(b)\right]^+ t_b(x)+\right.$$

$$\left.+\left[w(c)-\mathrm{J}_{\varepsilon_n} * \varphi(c)\right]^+ t_c(x)-\left[w(x)-\mathrm{J}_{\varepsilon_n} * \varphi(x)\right]^+\right|^2 dx+$$

(3) We have $\forall a,b\in R$ $|a^+-b^+|\leq|a-b|$.

$$+ \| h \|_{L^\infty(\Omega)} \int_{\Omega-\Omega_n} |w - J_{\varepsilon_n} * \varphi|^2 \, dx + 18 \| h \|_{L^\infty(\Omega)} \operatorname{meas}(\Omega'_n) \max_{x \in \overline{\Omega}}{}^2 | J_{\varepsilon_n} * \varphi (x) | \leq$$

$$\leq 72 \, l_n^2 \| h \|_{L^\infty(\Omega)} \operatorname{meas}(\Omega) \left(\max_{x \in \overline{\Omega}}{}^2 \left| \frac{\partial w}{\partial x_1}(x) \right| + \right.$$

$$\left. + \max_{x \in \overline{\Omega}}{}^2 \left| \frac{\partial w}{\partial x_2}(x) \right| + \| \varphi \|^2_{L^2(\Omega)} \, l_n^{-1} \left(\left\| \frac{\partial J}{\partial x_1} \right\|^2_{L^2(R^2)} + \left\| \frac{\partial J}{\partial x_2} \right\|^2_{L^2(R^2)} \right) \right) +$$

$$+ 2 \| h \|_{L^\infty(\Omega)} \max_{x \in \overline{\Omega}} w^2(x) \operatorname{meas}(\Omega - \Omega_n) + 2 \| h \|_{L^\infty(\Omega)} \int_{\Omega-\Omega n} (J_{\varepsilon_n} * \varphi)^2 \, dx +$$

$$+ 18 \operatorname{meas}^{1/2}(\Omega'_n) \| h \|_{L^\infty(\Omega)} \| \varphi \|^2_{L^2(\Omega)} \| J \|^2_{L^2(R^2)} .$$

As a consequence, because of Lemma 2, (2) and (3)

$$(12) \qquad \lim_{n \to +\infty} \left\| h^{1/2} \left[\sum_{i=1}^{m_n} (W_{ni} - \Phi_{ni})^+ g_{ni} \right] - h^{1/2} (w - J_{\varepsilon_n} * \varphi)^+ \right\|_{L^2(\Omega)} = 0 .$$

(11) and (12) imply the thesis.

Finally, let us prove

THEOREM 3. *It results*

$$\lim_{n \to +\infty} \| u_n - u \|_{H^1(\Omega)} = 0 .$$

Proof. We divide the proof into three parts.

As first step, we show that a subsequence of $\{u_n\}$ exists weakly convergent in $H^1(\Omega)$ towards an element u_0 of $H^1_0(\Omega)$. In fact, because of (1) and taking into account that, $\forall n \in N$, u_n is a solution of Problem 5′, $0 \in H_n$ and $E_{2n}(u_n) \geq 0$, we have

$$(13) \quad \forall n \in N \;\; c_1 \| u_n \|^2_{H^1(\Omega)} \leq E_{2n}(0) + \langle F, u_n \rangle \leq$$

$$\leq \frac{1}{2} \| h \|_{L^\infty(\Omega)} \int_\Omega \left[\sum_{i=1}^{m_n} \left(- \Phi_{ni} \right)^+ g_{ni}(x) \right]^2 dx + \| f \|_{L^2(\Omega)} \| u_n \|_{H^1(\Omega)} .$$

Now let $n \in N$. We observe that $\forall T \in \mathscr{T}_n$ and $\forall x \in T$, taking account of (7), (8) and (9), it results [2] [3]

$$\left| \sum_{i=1}^{m_n} \left(- \Phi_{ni} \right)^+ g_{ni}(x) - \left[- J_{\varepsilon_n} * \varphi (x) \right]^+ \right| = \left| \left[- J_{\varepsilon_n} * \varphi (a) \right]^+ t_a(x) + \right.$$

$$+\left[-J_{\varepsilon_n}*\varphi(b)\right]^+ t_b(x)+\left[-J_{\varepsilon_n}*\varphi(c)\right]^+ t_c(x)-\left[-J_{\varepsilon_n}*\varphi(x)\right]^+\Big(t_a(x)+$$

$$+t_b(x)+t_c(x)\Big)\Big|\le\left|J_{\varepsilon_n}*\varphi(a)-J_{\varepsilon_n}*\varphi(x)\right|+\left|J_{\varepsilon_n}*\varphi(b)-J_{\varepsilon_n}*\varphi(x)\right|+$$

$$+\left|J_{\varepsilon_n}*\varphi(c)-J_{\varepsilon_n}*\varphi(x)\right|\le 3\,l_n\left(\max_{x\in\overline{\Omega}}\left|\frac{\partial J_{\varepsilon_n}*\varphi}{\partial x_1}(x)\right|+\max_{x\in\overline{\Omega}}\left|\frac{\partial J_{\varepsilon_n}*\varphi}{\partial x_2}(x)\right|\right).$$

As a consequence, because of Lemma 2,

$$\int_\Omega\left|\sum_{i=1}^{m_n}\left(-\Phi_{ni}\right)^+ g_{ni}(x)-\left[-J_{\varepsilon_n}*\varphi(x)\right]^+\right|^2 dx=$$

$$=\sum_{T\in\mathscr{T}_n}\int_T\left|\sum_{i=1}^{m_n}\left(-\Phi_{ni}\right)^+ g_{ni}(x)-\left[-J_{\varepsilon_n}*\varphi(x)\right]^+\right|^2 dx+$$

$$+\int_{\Omega-\Omega_n}\left[-J_{\varepsilon_n}*\varphi(x)\right]^2 dx\le 36\,l_n\,\|\varphi\|^2_{L^2(\Omega)}\cdot\text{meas}(\Omega)\left(\left\|\frac{\partial J}{\partial x_1}\right\|^2_{L^2(R^2)}+\right.$$

$$\left.+\left\|\frac{\partial J}{\partial x_2}\right\|^2_{L^2(R^2)}\right)+\int_{\Omega-\Omega_n}\left(J_{\varepsilon_n}*\varphi\right)^2 dx+18\,\text{meas}^{1/2}(\Omega_n')\,\|\varphi\|^2_{L^2(\Omega)}\,\|J\|^2_{L^2(R^2)}\,;$$

thus, because of (2) and Lemma 2

$$\lim_{n\to+\infty}\left\|\sum_{i=1}^{m_n}(-\Phi_{ni})^+ g_{ni}-(-J_{\varepsilon_n}*\varphi)^+\right\|_{L^2(\Omega)}=0\,. \tag{14}$$

We observe now that from (5) it follows [7] $\lim_{n\to+\infty}\|(-J_{\varepsilon_n}*\varphi)^+-(-\varphi)^+\|_{L^2(\Omega)}=$ $=0$ which, with (14), implies

$$\lim_{n\to+\infty}\int_\Omega\left[\sum_{i=1}^{m_n}\left(-\Phi_{ni}\right)^+ g_{ni}\right]^2 dx=\int_\Omega\left[\left(-\varphi\right)^+\right]^2 dx\,. \tag{15}$$

Therefore, because of (13) and (15) a $c_7\in\,]0,+\infty[$ exists such that $\forall n\in N$ $\|u_n\|^2_{H^1(\Omega)}\le c_7+\|f\|_{L^2(\Omega)}\|u_n\|_{H^1(\Omega)}c_1^{-1}\le c_7+\|f\|^2_{L^2(\Omega)}c_1^{-2}/2+\|u_n\|^2_{H^1(\Omega)}/2$.

As a consequence

$$\exists\, c_8\in\,]0,+\infty[\,:\,\|u_n\|_{H^1(\Omega)}\le c_8\quad\forall n\in N\,. \tag{16}$$

(16) implies that it is possible to extract from $\{u_n\}$ a subsequence (which we indicate by the same index), weakly convergent in $H^1(\Omega)$ to an element $u_0\in$ $\in H_0^1(\Omega)$, i.e.

$$(17) \qquad u_n \rightharpoonup u_0 \quad \text{in } H^1(\Omega) .$$

As second step, we show now that

$$(18) \qquad u_0 = u .$$

Let $w \in C_0^\infty(\Omega)$, $n \in N$ and $r_n w = \sum_{i=1}^{m_n} W_{ni} g_{ni}$. Because u_n is solution of Problem 5′ and $r_n w \in H_0^1(\Omega)$ we have $a(u_n, u_n) + E_{2n}(u_n) \leq a(u_n, r_n w) + E_{2n}(r_n w) - \langle F, r_n w - u_n \rangle$. This result, taking into account that $0 \leq a(u_n - u_0, u_n - u_0) = a(u_n, u_n) + a(u_0, u_0) - 2a(u_n, u_0)$, implies

$$(19) \qquad \forall n \in N \quad -a(u_0, u_0) + 2a(u_n, u_0) + E_{2n}(u_n) \leq a(u_n, r_n w) + E_{2n}(r_n w) - \langle F, r_n w - u_n \rangle .$$

Now, let us notice that, since $w \in C_0^\infty(\Omega)$, it results [12]

$$(20) \qquad \lim_{n \to +\infty} \| r_n w - w \|_{H^1(\Omega)} = 0$$

from which, because of (17), $r_n w - u_n \rightharpoonup w - u_0$ in $H^1(\Omega)$. As a consequence

$$(21) \qquad \lim_{n \to +\infty} \langle F, r_n w - u_n \rangle = \langle F, w - u_0 \rangle .$$

Moreover from (17) it follows

$$(22) \qquad \lim_{n \to +\infty} a(u_n, u_0) = a(u_0, u_0) .$$

Furthermore,

$$\forall n \in N \quad | a(u_n, r_n w) - a(u_0, w) | \leq 2t \| u_n \|_{H^1(\Omega)} \| r_n w - w \|_{H^1(\Omega)} + | a(u_n - u_0, w) |$$

and thus, because of (17), (20) and (16)

$$(23) \qquad \lim_{n \to +\infty} a(u_n, r_n w) = a(u_0, w) .$$

Finally, we prove that

$$(24) \qquad \lim_{n \to +\infty}{}' \; E_{2n}(u_n) \geq \frac{1}{2} \int_\Omega h [(u_0 - \varphi)^+]^2 \, dx .$$

Obviously, since $\forall n \in N$

$$\sum_{i=1}^{m_n} \left(U_{ni} - \Phi_{ni} \right)^+ g_{ni}(x) \geq \left[\sum_{i=1}^{m_n} \left(U_{ni} - \Phi_{ni} \right) g_{ni}(x) \right]^+ \forall x \in \Omega,$$

we have

$$(25) \qquad \forall n \in \mathrm{N} \quad \mathrm{E}_{2n}(u_n) \geq \frac{1}{2}\int_\Omega h\left[(u_n - \varphi_n)^+\right]^2 \mathrm{d}x .$$

On the other hand, because of (17) and Lemma 3, we have $\lim\limits_{n\to+\infty} \| h^{1/2}(u_n - \varphi_n)^+ - h^{1/2}(u_0 - \varphi)^+ \|_{L^2(\Omega)} = 0$, from which $\lim\limits_{n\to+\infty} \frac{1}{2}\int_\Omega h\,[(u_n - \varphi_n)^+]^2\,\mathrm{d}x = \frac{1}{2}\int_\Omega h\,[(u_0 - \varphi)^+]^2\,\mathrm{d}x$. As a consequence, because of (25), (24) is true.

From (19), (21), (22), (23), (24) and Lemma 4 we have

$$(26) \qquad a(u_0, w - u_0) + \mathrm{E}_2(w) - \mathrm{E}_2(u_0) - \langle \mathrm{F}, w - u_0 \rangle \geq 0 \quad \forall w \in \mathrm{C}_0^\infty(\Omega) .$$

Since $\overline{\mathrm{C}_0^\infty(\Omega)} = \mathrm{H}_0^1(\Omega)$ and Problem 1 allows a unique solution, (26) implies (18).

As third step, we prove that $u_n \to u_0$ in $\mathrm{H}^1(\Omega)$. Let $n \in \mathrm{N}$. Since $\forall w \in \mathrm{C}_0^\infty(\Omega)\ a(u_n, u_n) + \mathrm{E}_{2n}(u_n) \leq a(u_n, r_n w) + \mathrm{E}_{2n}(r_n w) - \langle \mathrm{F}, r_n w - u_n \rangle$

we have

$$\mathrm{E}_{2n}(u_n) \leq c_1 \| u_n - u_0 \|^2_{\mathrm{H}^1(\Omega)} + \mathrm{E}_{2n}(u_n) \leq a(u_0, u_0) - 2\,a(u_n, u_0) + \\ + a(u_n, r_n w) + \mathrm{E}_{2n}(r_n w) - \langle \mathrm{F}, r_n w - u_n \rangle .$$

From this relation, taking account of Lemma 4 and that

$$| a(u_n, r_n w) - a(u_0, w) | \leq | a(u_n, r_n w) - a(u_n, w) | + | a(u_n, w) - \\ - a(u_0, w) | ,$$

we have

$$(27) \qquad \forall w \in \mathrm{C}_0^\infty(\Omega) \quad \lim_{n\to+\infty}{}' \ \mathrm{E}_{2n}(u_n) \leq \lim_{n\to+\infty}{}'' \ \mathrm{E}_{2n}(u_n) \leq - a(u_0, u_0) + \\ + a(u_0, w) + \mathrm{E}_2(w)$$

$$(28) \qquad \forall w \in \mathrm{C}_0^\infty(\Omega) \quad \lim_{n\to+\infty}{}' \ \mathrm{E}_{2n}(u_n) \leq \lim_{n\to+\infty}{}' \ (c_1 \| u_n - u_0 \|^2_{\mathrm{H}^1(\Omega)} + \\ + \mathrm{E}_{2n}(u_n)) \leq \lim_{n\to+\infty}{}'' \ (c_1 \| u_n - u_0 \|^2_{\mathrm{H}^1(\Omega)} + \mathrm{E}_{2n}(u_n)) \leq \\ \leq - a(u_0, u_0) + a(u_0, w) + \mathrm{E}_2(w) .$$

From (27) and (28), taking account of (24) and that $\overline{C_0^\infty(\Omega)} = H_0^1(\Omega)$ and $u_0 \in H_0^1(\Omega)$, we have $\lim\limits_{n\to+\infty} E_{2n}(u_n) = 0$, $\lim\limits_{n\to+\infty} (c_1 \| u_n - u_0 \|^2_{H^1(\Omega)} + E_2(u_n)) = E_2(u_0)$. Thus $\lim\limits_{n\to+\infty} \| u_n - u \|_{H^1(\Omega)} = 0$.

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