Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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A Weitzenbôck formula for the second fundamental form of a Riemannian foliation

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **77** (1984), n.3-4, p. 102–110.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1984_8_77_3-4_102_0>

Geometria differenziale. — *A Weitzenböck formula for the second fundamental form of a Riemannian foliation* (*). Nota (**) di Paolo Piccinni, presentata dal Socio E. Martinelli.

Riassunto. — Si considera la seconda forma fondamentale α di foliazioni su varietà riemanniane e si ottiene una formula per il laplaciano $\nabla^2 \alpha$. Se ne deducono alcune implicazioni per foliazioni su varietà a curvatura costante.

§ 1. Introduction

1. Let N be an n–dimensional sub–manifold of an m–dimensional Riemannian manifold M. Let ∇^{N} and ∇^{M} be the Levi Civita connections of N and M, R^{N} and R^{M} their curvature operators, and let B be the second fundamental form defined by:

$$\nabla^{M}_{X} Y = \nabla^{N}_{X} Y + B(X, Y) . \tag{1.1}$$

B satisfies in particular the Codazzi equation:

$$\pi (R^{M}(X, Y) Z) = (\nabla_{X} B)(Y, Z) - (\nabla_{Y} B)(X, Z) \tag{1.2}$$

(X, Y, Z vector fields on N, $\pi : TM \to Q$ projection in the normal bundle of N, and $(\nabla_X B)(Y, Z) = \pi \nabla^{M}_{X} B(Y, Z) - B(\nabla^{N}_{X} Y, Z) - B(Y, \nabla^{N}_{X} Z))$.

For any orthonormal basis $\{e_i\}$ of the tangent space T_pN and any further $x, y \in T_pN$ one can find orthonormal local extensions $\{E_i\}$ of $\{e_i\}$ and local extensions X, Y of x, y satisfying $\nabla^{N}_{e_i} E_j = \nabla^{N}_{e_i} X = \nabla^{N}_{e_i} Y = 0; i, j = 1, \dots, n$ (see for instance [8; § 1]). By using such vector fields and the Codazzi equation, J. Simons [7] obtained that the Laplacian $\nabla^2 B$ of B can be expressed by the following formula:

(*) Lavoro eseguito mentre l'Autore fruiva di una borsa di studio del C.N.R. per l'estero, presso la University of Illinois at Urbana-Champaign.

(**) Pervenuta all'Accademia il 22 ottobre 1984.

$$(\nabla^2 B)(x, y) = \sum_{i=1}^{n} (\nabla_{e_i} \nabla_{E_i} B)(x, y) =$$

$$\begin{aligned} &= \sum_{i=1}^{n} [D_{e_i}(\pi(R^M(E_i, X) Y)) + D_x(\pi(R^M(E_i, Y) E_i)) + \\ (1.3) \qquad &+ R_D(e_i, x) B(e_i, y) - B(R^N(e_i, x) e_i, y) - \\ &- B(e_i, R^N(e_i, x) y)] + D_x D_Y(\mathrm{Tr}\, B), \end{aligned}$$

where $D = \pi \nabla^M$ is the normal connection induced on Q.

Simons' formula 1.3 is contained also in the recent book [9], where several applications of it are given.

In this paper we obtain a formula of Simons' type in the case of a foliation $\mathscr{F}$ on M. We use the second fundamental form α of $\mathscr{F}$ as defined by F.W. Kamber and Ph. Tondeur ([2]). Since α is a symmetric bilinear form on all the tangent bundle of M (with values in the normal bundle Q of $\mathscr{F}$), our formula for the Laplacian $\nabla^2 \alpha$ of α applies not only to vectors which are tangent to the sub-manifolds that are leaves of the foliation, but to all the tangent vectors of M (formula 3.1). Nevertheless we remark that our formula 3.1, restricted to the leaves, does not give exactly Simons' formula 1.3, but a slightly different formula, since we make a different choice of the connections and of the vector fields that are involved.

Formulas giving local expressions of Laplace operators are usually referred to as Weitzenböck formulas (see, for instance, [3], [8]). In the case of a Riemannian foliation on a manifold with constant curvature we obtain also a "scalar" Weitzenböck formula (5.3), expressing the Laplacian of the square of the norm of the Weingarten operator associated to α. This scalar formula has several geometrical implications (7.1).

The author would like to thank Professor Philippe Tondeur for his encouragement and for many discussions on this subject.

§ 2. THE WEITZENBÖCK FORMULA FOR $\nabla^2 \alpha$

2. Let $\mathscr{F}$ be an n-dimensional foliation on an m-dimensional Riemannian manifold M. The metric g_M defines a splitting $\sigma : Q \to TM$ of the exact sequence

$$0 \to L \to TM \xrightarrow{\pi} Q \to 0,$$

where TM is the tangent bundle, L the integrable sub-bundle defining $\mathscr{F}$ and Q the quotient bundle of $\mathscr{F}$.

Consider on Q the connection ∇ defined by

$$\nabla_X s = \begin{cases} \pi [X, Y_s] & \text{if } X \in \Gamma L \\ \pi (\nabla_X^M Y_s) & \text{if } X \in \Gamma \sigma Q \end{cases}, \tag{2.1}$$

where ΓL and $\Gamma\sigma Q$ are the sets of the vector fields respectively tangent and normal to $\mathscr{F}$, and where we use the notation $Y_s = \sigma(s)$ for $s \in \Gamma Q$. From 2.1 we see that ∇ is the partial Bott connection along the leaves of $\mathscr{F}$, and it is induced by the Levi Civita connection ∇^M in the normal direction.

We recall that $\mathscr{F}$ is a *Riemannian foliation* if the metric g_Q induced on Q by g_M is parallel along the leaves with respect to the Bott connection. It follows that if $\mathscr{F}$ is Riemannian, the connection ∇ (2.1) is the only connection on Q that is compatible with g_Q and torsion free in the sense that $T_\nabla(X, Y) = \nabla_X \pi(Y) - \nabla_Y \pi(X) - \pi[X, Y] \equiv 0$ (cf. [2, p. 90] or [6, pp. 155-156]).

For any foliation $\mathscr{F}$ on M the symmetric bilinear form $\alpha : \Gamma TM \times \Gamma TM \to \Gamma Q$ defined by

$$\alpha(X, Y) = \pi(\nabla_X^M Y) - \nabla_X \pi(Y) \tag{2.2}$$

is called *second fundamental form of* $\mathscr{F}$ ([2, p. 94]) and reduces to the classical form B (1.1) on every leaf of $\mathscr{F}$.

The form α satisfies the following *Codazzi equation for foliations* ([3]):

$$\begin{aligned} \pi(R^M(X, Y) Z) - R_\nabla(X, Y)\pi(Z) &= \\ &= (\nabla_X \alpha)(Y, Z) - (\nabla_Y \alpha)(X, Z) \end{aligned} \tag{2.3}$$

$(X, Y, Z \in \Gamma TM)$, that restricted to any leaf gives the classical equation 1.2.

Fixed any $p \in M$ consider next an orthonormal basis $\{e_A\}$ of T_pM and an orthonormal local extension $\{E_A\}$ of $\{e_A\}$ such that $\nabla_{e_A}^M E_B = 0$ for all $A, B = 1, \ldots, m$; also for any further $x = \Sigma x_A e_A$, $y \in \Sigma y_A e_A$ in T_pM consider the local extensions $X = \Sigma x_A E_A$, $Y = \Sigma y_A E_A$ so that also $\nabla_{e_A}^M X = \nabla_{e_A}^M Y = 0$.

From equation 2.3 we obtain:

$$\begin{aligned} \sum_A (\nabla_{e_A} \alpha)(E_A, Y) &= \sum_A (\nabla_{e_A} \alpha)(Y, E_A) = \\ &= \sum_A [\pi(R^M(e_A, y) e_A) - R_\nabla(e_A, y)\pi(e_A)] + \operatorname{Tr}(\nabla_y \alpha). \end{aligned} \tag{2.4}$$

3. The following evaluation uses 2.3 and the conditions $\nabla^M_{e_A} E_B = \nabla^M_{e_A} X = \nabla^M_{e_A} Y = 0$:

$$(\nabla^2 \alpha)(x, y) = \sum_A \nabla_{e_A} ((\nabla_{E_A} \alpha)(X, Y)) =$$

$$= \sum_A [\nabla_{e_A} (\pi (R^M (E_A, X) Y)) - \nabla_{e_A} (R_\nabla (E_A, X) \pi (Y)) +$$

$$+ (R_\nabla (e_A, x) \alpha)(e_A, y) + \nabla_x ((\nabla_{E_A} \alpha)(E_A, Y))],$$

where $R_\nabla (e_A, x) \alpha = (\nabla_{e_A} \nabla_X - \nabla_x \nabla_{E_A} - \nabla_{[E_A, X]_p}) \alpha$ and $[E_A, X]_p = \nabla^M_{e_A} X - \nabla^M_x E_A = 0$. By applying equation 2.4 to the last term in the r.h.s. and by using the identity:

$$(R_\nabla (e_A, x) \alpha)(e_A, y) =$$

$$= R_\nabla (e_A, x) \alpha (e_A, y) - \alpha (R^M (e_A, x) e_A, y) - \alpha (e_A, R^M (e_A, x) y),$$

this yields:

(3.1) PROPOSITION. *The Laplacian* $\nabla^2 \alpha$ *is given by*:

$$(\nabla^2 \alpha)(x, y) =$$

$$= \sum_{A=1}^{m} [\nabla_{e_A} (\pi (R^M (E_A, X) Y)) - \nabla_{e_A} (R_\nabla (E_A, X) \pi (Y)) +$$

$$+ \nabla_x (\pi (R^M (E_A, Y) E_A) - \nabla_x (R_\nabla (E_A, Y) \pi (E_A)) +$$

$$+ R_\nabla (e_A, x) \alpha (e_A, y) - \alpha (R^M (e_A, x) e_A, y) - \alpha (e_A, R^M (e_A, x) y)] +$$

$$+ \nabla_x \nabla_Y \operatorname{Tr} \alpha .$$

If we choose the frame $\{e_A\}$ such that the first n vectors $e_1, \ldots, e_n$ are tangent to the leaf through p and consider the decomposition:

$$(\nabla^2 \alpha)(x, y) = \sum_{i=1}^{n} \nabla_{e_i} ((\nabla_{e_i} \alpha)(X, Y)) + \sum_{\beta = n+1}^{m} \nabla_{e_\beta} ((\nabla_{e_\beta} \alpha)(X, Y)), \tag{3.2}$$

from 3.1, 2.1 and our assumptions on the vector fields we see that:

$$\sum_{i=1}^{n} \nabla_{e_i} ((\nabla_{e_i} \alpha)(X, Y)) = \pi \nabla^M_{e_i} (R^M (E_i, X) Y) - \tag{3.3}$$

$$- \pi \nabla^M_x (R^M (E_i, Y) E_i) - \alpha (R^M (e_i, x) e_i, y) - \alpha (e_i, R^M (e_i, x) y) +$$

$$\nabla_x \nabla_Y \operatorname{Tr} \alpha ,$$

where x, y are tangent to the leaf. We note in particular that $\alpha (e_\gamma, e_\gamma) = 0$ for $\gamma = n + 1, \ldots, m$ and hence $\operatorname{Tr} \alpha = \sum_{i=1}^{n} \alpha (e_i, e_i)$.

Formula 3.3 gives an expression of the type 1.3 along the leaves, but with different choices of the connections and of the vector fields.

4. Assume now $M = M(k)$ be a manifold with constant sectional curvature k. Then the curvature operator of M is given by:

$$R^M(x, y)z = k[g_M(y, z)x - g_M(x, z)y],$$

and from def. 2.2 and our choices of X, Y we get:

$$\begin{aligned}
&\nabla_{e_A}(\pi(R^M(E_A, X)Y)) = k[-g_M(x, y)\alpha(e_A, e_A) + g_M(e_A, y)\alpha(e_A, x)] \\
(4.1)\quad &\nabla_x(\pi(R^M(E_A, Y)E_A)) = k[-g_M(e_A, y)\alpha(e_A, x) + \alpha(x, y)].
\end{aligned}$$

Formulas 4.1 allow us to simplify the r.h.s. of 3.1 for $M = M(k)$. In order to get a further simplification we make the assumption that the connection ∇ in the normal bundle Q of $\mathcal{F}$ is flat ($R_\nabla = 0$). This condition is equivalent to the possibility of extending any frame in the fiber Q_p by local sections in Q which are parallel with respect to ∇. This can be proved by the same argument used for sub-manifolds with flat normal connection (cf. for instance [1]).

(4.2) LEMMA. *Let* $M = M(k)$ *and let* $\mathcal{F}$ *be a foliation on* M *with flat connection* ∇. *Then the Laplacian* $\nabla^2\alpha$ *is given by*:

$$(\nabla^2\alpha)(x, y) = 2k[m\,\alpha(x, y) - g_M(x, y)\operatorname{Tr}\alpha] + \nabla_x\nabla_Y\operatorname{Tr}\alpha,$$

for any $x, y \in T_pM$.

Proof. From 3.1, by using equations 4.1 and $R_\nabla = 0$ one has easily:

$$\begin{aligned}
(\nabla^2\alpha)(x, y) = \sum_A k\{&-g_M(x, y)\alpha(e_A, e_A) + m\,\alpha(x, y) - \\
&-g_M(e_A, x)\alpha(e_A, y) + m\,\alpha(x, y) - g_M(x, y)\alpha(e_A, e_A) + \\
&+ g_M(e_A, y)\alpha(e_A, x)\} + \nabla_x\nabla_Y\operatorname{Tr}\alpha.
\end{aligned}$$

Also, if $x = \sum_A x_A e_A$, $y = \sum_A y_A e_A$, one obtains:

$$\begin{aligned}
&\sum_A\{-g_M(e_A, x)\alpha(e_A, y) + g_M(e_A, y)\alpha(e_A, x)\} = \\
&= \sum_{A,B}\{-x_A y_B\,\alpha(e_A, e_B) + x_B y_A\,\alpha(e_A, e_B)\} = 0,
\end{aligned}$$

and hence the claimed formula.

§ 3. THE SCALAR WEITZENBÖCK FORMULA

5. In the same context as in n. 4 consider for every normal vector $v \in Q_p$ the Weingarten operator $A_v : T_pM \to T_pM$ defined by:

$$g_M (A_v x , y) = g_Q (\alpha (x , y) , v)$$

and its Laplacian $(\nabla^2 A)_v : T_pM \to T_pM$ given by:

$$g_M ((\nabla^2 A)_v x , y) = g_Q ((\nabla^2 \alpha) (x , y) , v) . \tag{5.1}$$

In what follows, we assume to have chosen $\{e_A\}$ as indicated for formula 3.2.

(5.2) LEMMA. *With the same hypotheses of Lemma* 4.2, *we have*:

$$g_M (\nabla^2 A , A) = 2 k [m g_M (A , A) - g_Q (\mathrm{Tr}\, \alpha , \mathrm{Tr}\, \alpha)] +$$
$$+ \sum_{B , \beta} g_Q (\nabla_{e_B} \nabla_{\widetilde{A_\beta e_\beta}} \mathrm{Tr}\, \alpha , e_\beta) ,$$

where we used $A_\beta = A_{e_\beta}$, $\widetilde{A_\beta e_B}$ *local extension of* $A_\beta e_B$ *that is covariant constant at* p *with respect to* ∇^M, *and where we assumed* $B = 1 , \dots , m$; $\beta = n + 1 , \dots , m$.

Proof. From 5.1 and Lemma 4.2 we see that:

$$g_M (\nabla^2 A , A) = \sum_{B , \beta} g_M ((\nabla^2 A)_\beta e_B , A_\beta e_B) =$$
$$= \sum_{B , \beta} \{2 k m g_Q (\alpha (e_B , A_\beta e_B) , e_\beta) - 2 k g_M (e_B , A_\beta e_B) g_Q (\mathrm{Tr}\, \alpha , e_\beta) +$$
$$+ g_Q (\nabla_{e_B} \nabla_{\widetilde{A_\beta e_B}} \mathrm{Tr}\, \alpha , e_\beta)\} .$$

Since:

$$\sum_{B , \beta} g_Q (\alpha (e_B , A_\beta e_B) , e_\beta) = \sum_{B , \beta} g_M (A_\beta e_B , A_\beta e_B) = g_M (A , A)$$

and:

$$\sum_{B} g_M (e_B , A_\beta e_B) = \sum_{B} g_Q (\alpha (e_B , e_B) , e_\beta) = g_Q (\mathrm{Tr}\, \alpha , e_\beta) ,$$

we have the conclusion.

(5.3) THEOREM. *Let* $M = M (k)$ *be a manifold with constant curvature* k *and let* $\mathscr{F}$ *be a Riemannian foliation on* M *with flat connection* ∇. *Then the Laplacian of the square of the norm of the Weingarten operator* A *of* $\mathscr{F}$ *is given by*:

$$\frac{1}{2} \Delta g_M (A , A) = g_M (\nabla A , \nabla A) +$$
$$+ 2 k [m g_M (A , A) - g_Q (\mathrm{Tr}\, \alpha , \mathrm{Tr}\, \alpha)] + \sum_{B , \beta} g_Q (\nabla_{e_B} \nabla_{\widetilde{A_\beta e_\beta}} \mathrm{Tr}\, \alpha , e_\beta) .$$

Proof. Since $\mathcal{F}$ is Riemannian we have that $\nabla_X g_Q = 0$ for $X \in \Gamma L$. From the definition of ∇ it is easy to see that actually $\nabla_X g_Q = 0$ for all $X \in \Gamma T M$, i.e. ∇ is metric (cf. n. 2). By applying the metric condition twice, we get:

$$\frac{1}{2} \Delta g_M (A, A) = g_M (\nabla^2 A, A) + g_M (\nabla A, \nabla A)$$

and from Lemma 5.2 the conclusion.

6. The following evaluation of the terms in the brackets of formula 5.3 will be useful in the applications. We assume $B, C = 1, \ldots, m$; $i, j = 1, \ldots, n$; $\beta, \gamma = n+1, \ldots, m$. Then:

$$\begin{aligned} & m\, g_M (A, A) - g_Q (\mathrm{Tr}\, \alpha, \mathrm{Tr}\, \alpha) = \\ & = m \sum_{B,\beta} g_M (A_\beta e_B, A_\beta e_B) - \sum_{\beta} g_Q^2 (\mathrm{Tr}\, \alpha, e_\beta) = \\ & = m \sum_{B,C,\beta} g_M^2 (A_\beta e_B, e_C) - \sum_{B,\beta} g_M^2 (A_\beta e_B, e_B) = \\ & = m \sum_{\substack{B,C,\beta \\ B \neq C}} g_M^2 (A_\beta e_B, e_C) + (m-1) \sum_{B,\beta} g_M^2 (A_\beta e_B, e_B) . \end{aligned}$$

It follows:

(6.1) LEMMA. $m\, g_M (A, A) - g_Q (\mathrm{Tr}\, \alpha, \mathrm{Tr}\, \alpha) \geq 0$ *and the equality holds iff* $A = 0$.

§ 4. APPLICATIONS

7. From the definition of α one sees that $\alpha (x, y) = 0$ if both $x, y \in Q_p$. The *mean curvature* μ of $\mathcal{F}$ is then defined by $\mu = \frac{1}{n} \mathrm{Tr}\, \alpha$ ($n = \dim \mathcal{F}$). If we choose $\{e_A\}$ as specified for formula 3.2 we have:

$$\mu = \frac{1}{n} \sum_i \alpha (e_i, e_i) .$$

(7.1) COROLLARY. *Let* $M = M(k)$ *be a compact Riemannian manifold with constant curvature* k, $\mathcal{F}$ *a Riemannian foliation on* M *with flat connection* ∇. *Then*:

(i) *if* $k > 0$ *and the mean curvature* μ *is parallel with respect to* ∇, *then the second fundamental form* α *of* $\mathcal{F}$ *is identically zero.*

(ii) *if* $k = 0$ *and the mean curvature* μ *is parallel with respect to* ∇, *then the second fundamental form* α *is also parallel with respect to* ∇.

(iii) *if $k < 0$ and the second fundamental form α is parallel with respect to ∇, then α is identically zero.*

Proof.

(i) If $\nabla\mu = 0$ from 5.3 we have:

$$\frac{1}{2}\Delta g_M(A, A) = g_M(\nabla A, \nabla A) + 2k[m\, g_M(A, A) - g_Q(\mathrm{Tr}\,\alpha, \mathrm{Tr}\,\alpha)],$$

where by the assumption $k > 0$ and by 6.1 the r.h.s. is non-negative. It follows that $\Delta g_M(A, A) \geq 0$ and then the divergence theorem $\int_M \Delta g_M(A, A)\, dv = 0$ implies $\Delta g_M(A, A) = 0$. Therefore the r.h.s. is identically zero and Lemma 6.1 gives the conclusion.

(ii) If $\nabla\mu = 0$ and $k = 0$ we get the equation

$$\frac{1}{2}\Delta g_M(A, A) = g_M(\nabla A, \nabla A)$$

and the conclusion follows from the divergence theorem.

(iii) If $\nabla A = 0$ we have also $\nabla\mu = 0$ and hence:

$$\frac{1}{2}\Delta g_M(A, A) = 2k[m\, g_M(A, A) - g_Q(\mathrm{Tr}\,\alpha, \mathrm{Tr}\,\alpha)]$$

where by $k < 0$ and 6.1 the r.h.s. is non-positive. Then $\Delta g_M(A, A) \leq 0$ and the conclusion follows as in (i).

(7.2) *Remark.* We observe that in 7.1 (i) the assumptions $k > 0$, $R_\nabla = 0$ are non-compatible if codim $\mathcal{F} > 1$. This follows from the O'Neill formula for Riemannian submersions ([4]), that in terms of the sectional curvatures $R_\nabla(e_\beta, e_\gamma)$ and $R^M(e_\beta, e_\gamma)$ of ∇ and ∇^M ($e_\beta, e_\gamma \in Q_p$ and orthonormal) can be written as:

$$R_\nabla(e_\beta, e_\gamma) = R^M(e_\beta, e_\gamma) + \frac{3}{4}\,|\,\pi^\perp[E_\beta, E_\gamma]\,|_p^2,$$

where $\pi^\perp : TM \to L$ is the orthogonal projection (cf. [5], p. 213 or [3]).

(7.3) *Remark.* The conclusion $\alpha = 0$ contained in the statements (i) and (iii) of 7.1 can be proved to be equivalent to the twofold condition that all the leaves of $\mathcal{F}$ are totally geodesic in M and that the normal bundle of $\mathcal{F}$ is also integrable and totally geodesic (cf. [2], p. 97).

In the case $M = S^m$ (sphere) it is well-known that the only totally geodesic sub-manifolds are the great n-spheres (cf. [5], p. 105). By the previous remark it follows that statement 7.1 (i) gives the non-existence of foliations on the spheres with the required properties.

(7.4) *Remark.* For Riemannian foliations on a compact manifold the assumption $\nabla\nabla\mu = 0$ implies $\nabla\mu = 0$ and this implies $\mu = 0$.

To obtain the first implication note that the metric condition for ∇ gives the identity

$$\frac{1}{2}\,\Delta g_Q(\mathrm{Tr}\,\alpha\,,\,\mathrm{Tr}\,\alpha) = g_Q(\nabla\,\mathrm{Tr}\,\alpha\,,\,\nabla\,\mathrm{Tr}\,\alpha) + g_Q(\nabla^2\,\mathrm{Tr}\,\alpha\,,\,\mathrm{Tr}\,\alpha)\,,$$

and this, integrated over M, shows that if $\nabla\nabla\mu = 0$ then also $\nabla\mu = 0$.

The second implication follows from the formulas

$$\Delta\pi = d_\nabla\, d_\nabla^*\,\pi = d_\nabla\,\mathrm{Tr}\,\alpha = \nabla\,\mathrm{Tr}\,\alpha\,,$$

where d_∇^* is the adjoint of the exterior differential d_∇ of ∇ and where $\mathrm{Tr}\,\alpha$ is thought as a 0-form on M with values in Q (cf. [2], pp. 103-104). Hence $\nabla\mu = 0$ implies $\Delta\pi = 0$ and since M is compact that $d_\nabla^*\,\pi = \mathrm{Tr}\,\alpha = 0$ and hence $\mu = 0$.

Therefore the statements of 7.1 can be obtained also by observing that $\nabla\mu = 0$ implies $g_Q(\mathrm{Tr}\,\alpha\,,\,\mathrm{Tr}\,\alpha) = 0$ and by the divergence theorem.

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