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Anatoli Plička, Paolo Terenzi

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Analisi funzionale. — *On bibasic systems and a Retherford's problem.* Nota (*) di ANATOLI PLIČKO e PAOLO TERENZI, presentata dal Socio L. AMERIO.

RIASSUNTO. — Ogni spazio di Banach ha un sistema bibasico (x_n, f_n) normalizzato; inoltre ogni successione (x_n) uniformemente minimale appartiene ad un sistema biortogonale limitato (x_n, f_n), dove (f_n) è M-basica e normante.

§ 1. NOTATIONS AND DEFINITIONS

Let X be a Banach space, (x_n) a sequence of X, F a subset of X* (the dual of X), we use the following notations:

$[x_n] = \overline{\text{span}}\,(x_n)$, S (X) = the unit sphere of X, $F^{\perp} = \{x \in X\,; f(x) = 0$ for every f of F$\}$.

Let Y be a subset of X and let F be a subset of S (X*), we say that F K–*norms* Y if $\|x\| \leq K \sup\{|f(x)|\,; f \in F\}$ for every x of Y, where $1 \leq K < \infty$; in the same way we can say that a subset of S (X) K–norms a subset of X*. Let $(x_n) \subset X$ and $(f_n) \subset X^*$, we say that (x_n, f_n) is *biorthogonal* if

$$f_m(x_n) = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}, \qquad \text{for every } m \text{ and } n,$$

which is equivalent to say that (x_n) is *minimal*, that is $x_m \notin [x_n]_{n \neq m}$ for every m.

Let (x_n, f_n) be biorthogonal, with $[x_n] = X$, we say that

a) (x_n, f_n) is *bounded* if $(\|x_n\| \cdot \|f_n\|)$ is bounded, which is equivalent to say that (x_n) is *uniformly minimal*, that is $\inf_m \text{dist}\,(x_m/\|x_m\|, [x_n]_{n \neq m}) > 0$;

b) (x_n) is *M–basis* of X if $[f_n]^{\perp} = \{0\}$;

c) (x_n) is *norming M–basis* of X if S $([f_n])$ 1–norms X;

d) (x_n) is *basis* of X if $x = \sum_{n=1}^{\infty} f_n(x)\, x_n$ for every x of X.

We also say that (x_n) is *M–basic* (*basic*) if it is M–basis (basis) of $[x_n]$. Hence we say that (x_n, f_n) is *bibasic* (*M–bibasic*) if (x_n) and (f_n) are both basic (M–

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basic). A basic sequence (x_n) is said to be *asymptotically monotone* if, for every m,

$$\left\| \sum_{n=1}^{m} a_n x_n \right\| \leq K_m \left\| \sum_{n=1}^{m+p} a_n x_n \right\| \qquad \text{for every} \quad (a_n)_{n=1}^{m+p},$$

where $\lim_{m \to \infty} K_m = 1$; in particular (x_n) is said to be *monotone* if $K_m = 1$ for every m.

Finally we say that (x_n, f_n) is *normalized* if $\| x_n \| = \| f_n \| = 1$ for every n.

§ 2. PROBLEMS ON BIBASIC AND M–BIBASIC SYSTEMS

Banach proved ([1] p. 107, Th. 3; see also [7] p. 112, Th. 12.1) that

if (x_n) is basic of X and (x_n, f_n) is biorthogonal, then (f_n) is basic.

Retherford (1964) raised the following problem (see also [5], Probl. 3.2)

"... If $Y \subset X$, X a Banach space and (y_n) a basis for Y, with coefficient functionals $g_n \in Y^*$. Does there exist a Hahn-Banach extension (f_n) of (g_n) in X^* such that (f_n) is a basic sequence in X? ...".

If the Hahn-Banach extension f_n of g_n is without conditions on the norm, this problem has a positive answer ([8] p. 84, Pbl. 1.6; see also p. 856).

Instead, if the extension is with the same norm, the next example gives a negative answer, also with the weaker condition of (f_n) M–basic and also if Y has codimension one in X.

Example. Let $X = l^1$, $(e_n)_{n \geq 0}$ the natural basis of l^1, set

(1) $$y_n = (e_n - e_0) / 2 \qquad \text{for every} \quad n \geq 1 \, , \; Y = [y_n]_{n \geq 1} \, .$$

There exists (g_n) of Y^* with (y_n, g_n) biorthogonal and normalized; however there exists a unique extension with the same norm (f_n) of (g_n) in X^*, which is not M–basic.

From the above a natural question arises as to whether the extension with a bounded norm is possible, that is if there is a positive answer for the intermediate case. Precisely we have

Problem 1. Does every basic sequence belong to a bounded bibasic system?

Problem 2. Does every uniformly minimal M–basic sequence belong to a bounded M-bibasic system?

The next theorem gives a positive answer to Problem 2.

THEOREM I. *Every uniformly minimal sequence (x_n) of X belongs to a bounded biorthogonal system (x_n, f_n), where (f_n) is norming M-basic.*

Recall that the existence of bounded bibasic systems was stated in [2], moreover in [9] (Cor. I*, p. 352) we stated the existence of bibasic systems (x_n, f_n) with $\| x_n \| \cdot \| f_n \| < 1 + \varepsilon$ for every n, for every fixed $\varepsilon > 0$. Hence in [9] the following question was raised:

Problem 3. Does there exist a bibasic system (x_n, f_n) normalized?

Next theorem answers Problem 3.

THEOREM II. *Every Banach space has a normalized bibasic system (x_n, f_n) with (x_n) and (f_n) asymptotically monotone.*

Problem 1 is still open.

§ 3. PROOFS

Proof of example. (y_n) is basic monotone, indeed by (1) for every $(a_n)_{n=1}^{m+p}$ it follows that

$$\left\| \sum_{n=1}^{m} a_n y_n \right\| = \left\| \left(- \sum_{n=1}^{m} \frac{a_n}{2} \right) e_0 + \sum_{n=1}^{m} \frac{a_n}{2} e_n \right\| = \left| \sum_{n=1}^{m} \frac{a_n}{2} \right| + \sum_{n=1}^{m} \frac{|a_n|}{2} =$$

$$= \left| \sum_{n=1}^{m+p} \frac{a_n}{2} - \left(\sum_{n=m+1}^{m+p} \frac{a_n}{2} \right) \right| + \sum_{n=1}^{m} \frac{|a_n|}{2} \leq \left| \sum_{n=1}^{m+p} \frac{a_n}{2} \right| + \sum_{n=1}^{m+p} \frac{|a_n|}{2} = \left\| \sum_{n=1}^{m+p} a_n y_n \right\| .$$

Moreover $\operatorname{dist}(y_m, [y_n]_{n \neq m}) = \| y_m \| = 1$ for every m; indeed by (1) for every m and for every $(a_n)_{n=1, n\neq m}^{p}$ it follows that

$$\left\| y_m + \sum_{n=1, n\neq m}^{p} a_n y_n \right\| = \left\| - \left(1 + \sum_{n=1, n\neq m}^{p} a_n \right) \frac{e_0}{2} + \frac{e_m}{2} + \sum_{n=1, n\neq m}^{p} \frac{a_n}{2} e_n \right\| =$$

$$= \left(\left| 1 + \sum_{n=1, n\neq m}^{p} a_n \right| + 1 + \sum_{n=1, n\neq m}^{p} |a_n| \right) \frac{1}{2} \geq \left| 1 + \sum_{n=1, n\neq m}^{p} a_n - \right.$$

$$\left. - \sum_{n=1, n\neq m}^{p} a_n \right| \frac{1}{2} + \frac{1}{2} = 1 .$$

Therefore there exists (g_n) of Y^* such that

(2) $\qquad (y_n, g_n)_{n\geq 1}$ is biorthogonal and normalized.

Fix a natural number m.

Let $f_m \in X^*$ such that

(3) $\qquad f_m$ is extension of g_m with the same norm, $f_m = (a_{mn})_{n=0}^{\infty}$.

Now $f_m(y_m) = 1$ implies $(a_{mm} - a_{m0})/2 = 1$, while $f_m(y_n) = 0$ implies $(a_{mn} - a_{m0})/2 = 0$ for every $n \neq m$. On the other hand by (2) and (3) $\sup_n |a_{mn}| = 1$, hence it follows that

$$a_{mm} = 1 \quad , \quad a_{m0} = -1 \quad , \quad a_{mn} = a_{m0} \qquad \text{for} \quad n \neq m\,. \tag{4}$$

Therefore the Hahn-Banach extension (f_n) of (g_n) in X*, with the same norm, is unique. Let $\bar{f} \in X^*$ such that

$$\bar{f} = (\bar{a}_n) \quad , \qquad \text{with} \quad \bar{a}_n = 1 \qquad \text{for every} \quad n \geq 0\,. \tag{5}$$

We affirm that

$$\bar{f} \in \bigcap_{m=1}^{\infty} [f_n]_{n \geq m}\,. \tag{6}$$

Indeed fix m.

For every p and k by (5) and (4) we have that

$$\left(\bar{f} + \frac{1}{p} \sum_{n=m+1}^{m+p} f_n\right)(e_k) = 1 + \frac{1}{p} \sum_{n=m+1}^{m+p} f_n(e_k)\,;$$

where

$$\frac{1}{p} \sum_{n=m+1}^{m+p} f_n(e_k) = \begin{cases} -1 \text{ if } k \leq m & \text{and if} \quad k \geq m+p+1\,, \\ -1 + \dfrac{2}{p} \text{ if } m+1 \leq k \leq m+p\,, \end{cases}$$

hence

$$\left\| \bar{f} + \frac{1}{p} \sum_{n=m+1}^{m+p} f_n \right\| = \sup_k \left| \left(\bar{f} + \frac{1}{p} \sum_{n=m+1}^{m+p} f_n\right)(e_k) \right| = \frac{2}{p}\,;$$

consequently

$$\lim_{p \to \infty} \left\| \bar{f} + \frac{1}{p} \sum_{n=m+1}^{m+p} f_n \right\| = 0\,.$$

Therefore (6) is proved, hence (f_n) is not M–basic ([8] p. 225, Rem. 8.2); which completes proof of the example.

Proof of Theorem I. We can suppose

$$(x_n\,, g_n) \text{ biorthogonal}\,, \ \|x_n\| = 1 \text{ and } \|g_n\| \leq K < \infty \text{ for every } n\,. \tag{7}$$

Set $f_1 = g_1\,, f_2 = g_2$ and proceed by induction.

Fix $m \geq 2$.

Suppose that we have $(f_n)_{n=1}^{m} \cup (g_{mn})_{n>m}$ of X^*, moreover (only if $m > 2$) a sequence $(z_n)_{n=1}^{p_{m-1}}$ of X and two sequences $(p_n)_{n=2}^{m-1}$ and $(q_n)_{n=2}^{m-1}$ of natural numbers, so that

$$(8) \quad \begin{aligned} & (x_n , f_n)_{n=1}^{m} \cup (x_n , g_{mn})_{n>m} \text{ is biorthogonal ;} \\ & \| f_n \| < 3\,K \text{ for } 1 \leq n \leq m \text{ and } \| g_{mn} \| < 3\,K \text{ for } n > m \, ; \\ & (z_n)_{n=1}^{p_{m-1}} \subset [(f_n)_{n=1}^{m} \cup (g_{mn})_{n>m}]^{\perp} \\ & \text{moreover for every } n, \text{ with } 2 \leq n \leq m-1 \, , \\ & [z_k]_{k=1}^{p_n} + [x_k]_{k=1}^{q_n} \left(1 + \frac{1}{2^n}\right) \text{— norms } [f_k]_{k=1}^{n} \, . \end{aligned}$$

There exist a natural number q'_m and a sequence $(y_n)_{n=p_{m-1}+1}^{p_m}$ of X so that

$$[x_n]_{n=1}^{q'_m} + [z_n]_{n=1}^{p_{m-1}} + [y_n]_{n=p_{m-1}+1}^{p_m} \left(1 + \frac{1}{2^m}\right) \text{— norms } [f_n]_{n=1}^{m} \, .$$

By (7) and by [10] (p. 502, Lemma 1) there exist a natural number t_{m+1} and a sequence $(g_{m+1,n})_{n>t_{m+1}}$ of X^*, so that

$$t_{m+1} \geq m+1 \, , \, (x_n , g_{m+1,n})_{n>t_{m+1}} \quad \text{is biorthogonal ;}$$

$$[x_n]_{n=1}^{t_{m+1}} + [z_n]_{n=1}^{p_{m+1}} + [y_n]_{n=p_{m-1}+1}^{p_m} \subset [(g_{m+1,n})_{n>t_{m+1}}]^{\perp} ;$$

$$\| g_{m+1,n} \| < 3\,K \text{ for } n > t_{m+1} \, .$$

Set

$$f_{m+1} = g_{m,m+1} \, , \, g_{m+1,n} = g_{m,n} \qquad \text{for} \quad m+2 \leq n \leq t_{m+1} \, ;$$

$$z_n = y_n - \left(\sum_{k=1}^{m+1} f_k(y_n)\, x_k + \sum_{k=m+2}^{t_{m+1}} g_{m+1,k}(y_n)\, x_k \right) \quad \text{for} \quad p_{m-1} + 1 \leq n \leq p_m .$$

Hence, setting $q_m = \max \{q'_m , t_{m+1}\}$, we have (8) with $m+1$ instead of m.

So proceeding we get (f_n) of X^*, (z_n) of X and two sequences $(p_n)_{n\geq 2}$ and $(q_n)_{n\geq 2}$ of natural numbers, so that

$$(9) \quad \begin{aligned} & (x_n , f_n) \text{ is biorthogonal, } (z_n) \subset [f_n]^{\perp} \text{ and } \| f_n \| < 3\,K \text{ for every } n \, ; \\ & [z_n]_{n=1}^{p_m} + [x_n]_{n=1}^{q_m} \left(1 + \frac{1}{2^m}\right) \text{— norms } [f_n]_{n=1}^{m} \text{ for every } m \geq 2 \, . \end{aligned}$$

Fix m and set

$$(10) \qquad F_m = [f_n]_{n=1}^{m} \quad , \quad F^m = [f_n]_{n>q_m} \quad , \quad Y_m = [x_n]_{n=1}^{q_m} + [z_n]_{n=1}^{p_m} .$$

Let P_m be the projector

$$(11) \qquad P_m : F_m + F^m \to F_m .$$

By (9) and (10) $F^m \subset Y_m^{\perp}$, moreover $Y_m (1 + 1/2^m)$–norms F_m, hence it is easy to see that

$$(12) \qquad \| P_m \| \leq 1 + \frac{1}{2^m} .$$

By (11) and (12) it follows that

$$\sup_m \operatorname{dist} \left(f , [f_n]_{n>m} \right) = \| f \| , \qquad \text{for every } f \text{ of } [f_n] .$$

Hence (f_n) is norming ([4], p. 121-122 and Lemma I.11); which completes the proof of Th. I.

Proof of Theorem II. We construct two sequences (x_n) of X and (f_n) of X^*, moreover two sequences of finite subsets (Y_n) of X and (G_n) of X^*, so that for every n:

$$(13) \qquad \begin{gathered} f_n (x_n) = 1 \quad , \quad x_n \in Y_n \subset S(X) \quad , \quad f_n \in G_n \subset S(X^*) \quad , \quad Y_{n-1} \subset Y_n \quad , \\ G_{n-1} \subset G_n \; ; Y_n \left(1 + \frac{1}{2^n}\right) \text{— norms } \Big[f_k \Big]_{k=1}^{n} \quad , \quad G_n \left(1 + \frac{1}{2^n} \right) \text{— norms} \\ [x_k]_{k=1}^{n} \; ; \; x_{n+1} \in G_n^{\perp} \text{ and } f_{n+1} \in Y_n^{\perp} . \end{gathered}$$

We pick x_1 of $S(X)$ and f_1 of $S(X^*)$ with $f_1(x_1) = 1$; let $Y_1 = x_1$ and $G_1 = f_1$.

If for $n - 1$ such objects are constructed, by Krasnoselski-Krein-Milman th. [3] (see also [6] p. 269) pick an element x_n of $S(G_{n-1}^{\perp})$, which is orthogonal for $[Y_{n-1}]$; and by Hahn-Banach theorem pick f_n of $S(X^*)$ such that $f_n(x_n) = 1$ and $Y_{n-1} \subset f_n^{\perp}$.

After, pick finite sets $Y_n \supset (x_n , Y_{n-1})$ and $G_n \supset (f_n , G_{n-1})$, which $(1 + 1/2^n)$–norm $[f_k]_{k=1}^{n}$ and $[x_k]_{k=1}^{n}$ respectively.

Fix m and set

$$[14] \qquad X_m = [x_n]_{n=1}^{m} \quad , \quad X^m = [x_n]_{n>m} \quad , \quad F_m = [f_n]_{n=1}^{m} \quad , \quad F = [f_n]_{n>m} .$$

Let P_m and Q_m be the projectors

$$P_m \ : \ X_m + X^m \to X_m \quad , \quad Q_m \ : \ F_m + F^m \to F_m \, .$$

By (13) and (14) $X^m \subset G_m^{\perp}$ and $F^m \subset Y_m^{\perp}$, where G_m $(1 + 1/2^m)$ – norms X_m and Y_m $(1 + 1/2^m)$–norms F_m; therefore it is easy to see that

$$\left\| P_m \right\| \leq 1 + \frac{1}{2^m} \quad , \quad \left\| Q_m \right\| \leq 1 + \frac{1}{2^m}\Big) \, .$$

Hence (x_n) and (f_n) are asymptotically monotone; which completes proof of Theorem II.

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