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Dean A. Carlson

On the Existence of Optimal Solutions for Infinite Horizon Optimal Control Problems: Nonconvex and Multicriteria Problems

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RIASSUNTO. — In questa Nota si continua la discussione iniziata in [4] dell'esistenza di soluzioni ottimali per problemi di ottimo controllo in $[0 + \infty]$. Si definiscono problemi generalizzati, e si ottengono estensioni di risultati già presentati in [4]. Si dimostrano anche varie relazioni tra le soluzioni ottimali dei problemi generalizzati e i problemi originali e non convessi di ottimo controllo. Alla fine si considerano problemi lineari nelle variabili di stato anche nel caso di costi funzionali a valori vettoriali (ottimizzazione alla Pareto).

1. INTRODUCTION

In this note we continue discussing the existence of optimal solutions, under minimal convexity hypotheses, for optimal control problems defined on $[0, +\infty)$. Our emphasis concentrates on the case when the cost functional, given as an improper integral, does not necessarily converge. This leads to a variety of weaker notions of optimality. Here we discuss extensions of the results presented in Carlson [4] to the case of nonconvex problems through the introduction of chattering states as well as to the case of a vector valued cost criteria (i.e., Pareto optimality).

2. THE MODEL

The model we treat is a Lagrange-type problem of optimal control. Specifically we consider a control system of the form

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad \text{a.e. on} \quad [0, +\infty) \tag{1}$$

$$x(0) = x_0 \tag{2}$$

where the state variable $x(t) \in E^n$ and the control variable $u(t) \in E^m$. Let $A = [0, +\infty) \times X \subseteq E^{1+n}$ be closed and for each $(t, x) \in A$ let $U(t, x) \subseteq E^m$ be such that the set $M = \{(t, x, u) : (t, x) \in A, u \in U(t, x)\}$ is closed. The function $f : M \to E^n$ is assumed to satisfy the usual Carathéodory conditions.

(*) Assistant Professor, Department of Mathematics and Statistics, University of Missouri-Rolla, Rolla, Missouri, 65401.

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We consider admissible pairs of functions $\{x , u\}$, each pair consisting of a measurable function $u : [0 , +\infty) \to E^m$ and a corresponding locally absolutely continuous function $x : [0 , +\infty) \to E^n$, which satisfies the control system (1)-(2) as well as the constraints

$$(t , x(t)) \in A , \qquad \text{for all} \quad t \geq 0 \tag{3}$$

$$u(t) \in U(t , x(t)) , \qquad \text{a.e. on} \quad [0 , +\infty) . \tag{4}$$

The objective functional we consider is described as an improper integral

$$J[x , u] = \int_0^{+\infty} g(t , x(t) , u(t)) \, dt , \tag{5}$$

where $g : M \to E^1$ is a Lebesgue normal integrand in the sense of Rockafellar [8]. Further we require an admissible pair $\{x , u\}$ to be such that the map $t \to g(t , x(t) , u(t))$ is locally Lebesgue integrable on $[0 , +\infty)$. The set of all admissible pairs, denoted by Ω, is assumed to be nonempty.

The hypotheses given above are insufficient to insure the convergence of the functional J for a given $\{x , u\} \in \Omega$ and thus the traditional notion of a minimum is not applicable. Therefore we deal with a variety of weaker notions of optimality that have recently arisen in the literature. Here however we restrict our attention to the case of a true minimum, referring the reader to Carlson [5] for a discussion of the corresponding results for the weaker types of optimality.

The necessary compactness property for the set of admissible trajectories is ensured by assuming that the functions f and g satisfy an extension of a standard growth condition utilized by Cesari [7] and others. Specifically, we say that f and g satisfy the *growth condition* (γ) if for every $\varepsilon > 0$ and $T > 0$ there exists $L_T \geq 0$ and an integrable function $\psi_{\varepsilon T} : [0 , +\infty) \to E^1$ such that $g(t , x , u) \leq L_T$ and $|f(t , x , u)| \leq \psi_{\varepsilon T}(t) + \varepsilon g(t , x , u)$ a.e. in $t \in [0 , T]$, $(t , x , u) \in M$. This growth condition ensures that the set of admissible trajectories is a relatively weak compact subset of the space of locally absolutely continuous functions, where here the topology is convergence of initial values and weak L^1-convergence of derivatives on compact subsets of $[0 , +\infty)$. For details see Carlson [5; 2.2].

3. Nonconvex Problems

A crucial assumption placed on the models in [4] is the convexity of the sets $\tilde{Q}(t , x) = \{(z^0 , z) : z^0 \geq g(t , x , u) , \; z = f(t , x , u) , \; u \in U(t , x)\}$. In this section we investigate the case when this convexity hypothesis does not hold. To discuss such problems we define the relaxed problem by defining the functions G and F as

$$\mathrm{G}(t, x, \boldsymbol{u}, \tilde{p}) = \sum_{k=1}^{n+2} p_k g(t, x, u_k) \tag{6}$$

and

$$\mathrm{F}(t, x, \tilde{u}, \tilde{p}) = \sum_{k=1}^{n+2} p_k f(t, x, u_k), \tag{7}$$

where for $k = 1, 2, \ldots, n+2$, $(t, x, u_k) \in \mathrm{M}$, $p_k \geq 0$, and $\sum_{k=1}^{n+2} p_k = 1$. Specifically we consider triples $\{x(t), \tilde{u}(t), \tilde{p}(t)\}$; where $x \in \mathrm{AC}_{\mathrm{loc}}$, $\tilde{p}(t) = (p_1(t), p_2(t), \ldots, p_{n+2}(t)) \in \mathrm{E}^{n+2}$ is measurable, and $\tilde{u}(t) = (u_1(t), u_2(t), \ldots u_{n+2}(t)) \in \mathrm{E}^{m\,(n+2)}$ is measurable; satisfying the control system $\dot{x}(t) = \mathrm{F}(t, x(t), \tilde{u}(t), \tilde{p}(t))$ a.e. on $[0, +\infty)$, $x(t) = x_0$, $(t, x(t)) \in \mathrm{A}$ on $[0, +\infty)$, for $k = 1, 2, \ldots, n+2$ $\tilde{u}_k(t) \in \mathrm{U}(t, x(t))$ a.e. on $[0, +\infty)$, and $\tilde{p}(t) \in \mathrm{P}$ a.e. on $[0, +\infty)$ where $\mathrm{P} = \left\{(p_1, p_2, \ldots, p_{n+2}) : p_k \geq 0, \sum_{k=1}^{n+2} p_k = 1\right\}$ and such that the map $t \to \mathrm{G}(t, x(t), \tilde{u}(t), \tilde{p}(t))$ is locally Lebesgue integrable. This set of triples we denote by $\tilde{\Omega}$ and refer to it as the set of relaxed admissible pairs.

The relaxed optimal control problem then consists of minimizing the functional

$$\mathrm{J}[x, u, p] = \int_0^{+\infty} \mathrm{G}(t, x(t), \tilde{u}(t), \tilde{p}(t))\, dt, \tag{8}$$

over the set $\tilde{\Omega}$. Formally this problem is identical with the original problem, except now the requisite convexity hypotheses are satisfied. That is, for $(t, x) \in \mathrm{A}$, the sets $\tilde{\tilde{\mathrm{Q}}}(t, x) = \{(z^0, z) : z^0 \geq \mathrm{G}(t, x, \tilde{u}, \tilde{p}),\ z = \mathrm{F}(t, x, \tilde{u}, \tilde{p}),\ u_k \in \mathrm{U}(t, x),\ k = 1, 2, \ldots, n+2,\ \tilde{p} \in \tilde{\mathrm{P}}\}$ are convex and in fact equal the convex hulls of $\tilde{\mathrm{Q}}(t, x)$. As a consequence of this, existence results for the relaxed problem are easily obtained from their corresponding non-relaxed counterparts. As a representative example we give the following theorem.

THEOREM 1. *Let* $\mathrm{A}, \mathrm{U}, \mathrm{M}, g$, *and* f *satisfy the hypotheses of Section* 2. *Further assume* (*i*). *there exists* $\alpha \in \mathrm{E}^1$ *such that* $\tilde{\Omega}_\alpha = \{\{x, \tilde{u}, \tilde{p}\} \in \tilde{\Omega} : \tilde{\mathrm{J}}[x, \tilde{u}, \tilde{p}] \leq \alpha\} \neq \varnothing$; (ii). *the growth conditon* (γ) *holds*; (*iii*). *there exists an integrable function* $\psi : [0, +\infty) \to \mathrm{E}^1$ *such that for all* $\{x, \tilde{u}, \tilde{p}\} \in \tilde{\Omega}_\alpha$, $\mathrm{G}(t, x(t), \tilde{u}(t), \tilde{p}(t)) \geq \psi(t)$ *a.e. on* $[0, +\infty)$; *and* (iv). *the sets* $\tilde{\tilde{\mathrm{Q}}}(t, x)$ *satisfy the upper semicontinuity condition property* (K)

$$\tilde{\tilde{\mathrm{Q}}}(t, x) = \bigcap_{\delta > 0} cl\left[\bigcup\{\tilde{\tilde{\mathrm{Q}}}(t, y) : |x - y| < \delta, (t, x) \in \mathrm{A}\}\right]. \tag{9}$$

Then the relaxed optimal control problem has an optimal solution.

It is well known that the set of admissible pairs Ω can be viewed as a subset of the relaxed admissible pairs $\tilde{\Omega}$. Thus, the infimum (when it exists) for the nonconvex control problem is larger than the infimum for the corresponding relaxed problem. To investigate the relationship between the relaxed admissible trajectories and the non-relaxed ones we give the following uniform approximation theorem which generalizes the corresponding finite horizon result (see Cesari [6; 18.6-18.7]). For this result we must strengthen some of the hypotheses given in Section 2. Most notably, we require the control set $U(t, x)$ to be independent of x (i.e., $U(t, x) = U(t)$ for all $x \in X$) and that there exists a bounded set $V \subset E^m$ such that $U(t) \subseteq V$ a.e. $t \geq 0$.

THEOREM 2. *Let $X \subseteq E^n$ be compact and let $A = [0, +\infty) \times X$. Let $U : [0, +\infty) \to 2^V$ be a measurable closed set valued map such that the set $M = \{(t, x, u) : (t, x) \in A, u \in U(t)\}$ is closed. Let $f : M \to E^n$ be a given continuous function and suppose there exists locally integrable functions m and k from $[0, +\infty)$ into E^1 such that $|f(t, x, u)| \leq m(t)$ and $|f(t, x, u) - f(t, z, u)| \leq k(t)|x - z|$ a.e. in $[0, +\infty)$, (t, x, u), $(t, z, u) \in M$. Let $\{x, \tilde{u}, \tilde{p}\}$ denote a relaxed admissible pair for the corresponding relaxed control system such that for all $t \geq 0$, $x(t) \in X_1$ where $X_1 \subseteq \operatorname{int}(X)$ is compact. Then there exists a sequence of admissible pairs; $\{x_n, u_n\}$, defined on $[0, n]$; for the original system such that $x_n \to x$ uniformly on $[0, T]$ for all $T > 0$.*

This result is established by applying the corresponding finite horizon result on $[0, n]$, for each positive integer n, to obtain a sequence $\{x_\varepsilon^n, u_\varepsilon^n\}$ of admissible pairs defined on $[0, n]$ approximating x.

When the above result is applied to the augmented control system $(\dot{x}^0(t), \dot{x}(t)) = (g(t, x(t), u(t)), f(t, x(t), u(t)))$, a.e. on $[0, +\infty)$, one obtains relationships between the optimal solutions of the original optimal control problem and those of the corresponding relaxed optimal control problem (see [5; 4.2]).

4. PROBLEMS WHICH ARE LINEAR IN THE STATE

Another class of nonconvex problems is the restrictive case where g and f are linear in the state variables. That is, the admissible pairs $\{x, u\} \in \Omega$ satisfy a control system of the form

$$\begin{aligned} \dot{x}(t) &= C(t)\, x(t) + f(t, u(t)) \quad , \quad \text{a.e. on} \quad [0, +\infty) \\ x(0) &= x_0 \end{aligned} \tag{11}$$

as well as the control constraints

$$u(t) \in U(t), \qquad \text{a.e. on} \quad [0, +\infty),$$

where $C : [0, +\infty) \to E^{n^2}$ is an $n \times n$ matrix whose entries are locally Lebesgue integrable, $U : [0, +\infty) \to 2^{E^m}$ is a set valued map with closed graph $M = \{(t, u) : t \geq 0, u \in U(t)\}$, and $f : M \to E^n$ satisfies the Carathéodory conditions. The cost functional in this case is an improper integral of the form

$$\text{(12)} \qquad \mathrm{J}[x, u] = \int_0^{+\infty} [\langle c_0(t), x(t) \rangle + g(t, u(t))] \, dt,$$

where $c_0 : [0, +\infty) \to \mathrm{E}^n$ is locally integrable, $g : \mathrm{M} \to \mathrm{E}^1$ is a Lebesgue normal integrand, and where $\langle \cdot, \cdot \rangle$ denotes the usual euclidean inner product on E^n.

For models of this type it is possible to establish a Neustadt-type existence theorem for strongly optimal solutions without any *a priori* convexity conditions, provided certain integrability conditons are placed on the entries of the functions c_0 and C insuring that the optimal trajectories have asymptotic equilibria and that the corresponding value of the functional J, given by (12) is finite. The proof follows classical lines (see e.g., Cesari [6 : 16.5i] and utilizes a recent extension to σ-finite measure spaces of a Lyapounov-type theorem concerning vector valued integrals found in Arstein [1]. We now state this existence theorem.

THEOREM 3. *In addition to the above hypotheses assume that (i). the entries of the functions* C *and* c_0 *are Lebesgue integrable on* $[0, +\infty)$; *(ii). there exists a locally Lebesgue integrable function such that* $|(g(t, u), f(t, u))| \leq m(t)$ *a.e. in* $t \geq 0$, $(t, u) \in \mathrm{M}$ *and such that*

$$\text{(13)} \qquad \int_0^{+\infty} \left[m(t) \exp\left(- \int_0^t tr(\mathrm{C}(s)) \, ds \right) dt \right] < +\infty,$$

where $tr(\mathrm{C})$ *denotes the trace of the matrix* C; *(iii). for* $t \geq 0$ *and* $x \in \mathrm{E}^n$ *the sets* $\tilde{\tilde{\mathrm{Q}}}(t, x)$ *for the corresponding relaxed optimal control problem satisfy property* (K); *and (iv). there exists a Lebesgue integrable function* $\psi \in \mathrm{L}([0, +\infty))$ *such that*

$$\langle c_0(t), x \rangle + g(t, u) > \psi(t), \qquad \text{a.e. } in \quad t \geq 0,$$

$(t, u) \in \mathrm{M}$ *and* $x \in \mathrm{E}^n$. *Then there exists an optimal solution for the linear in the state infinite horizon optimal control problem.*

Extension of the above result to the weaker notions of optimality listed in [4] are discussed in Carlson [5]. Also in [5] the integrability conditions placed on the functions C and c_0 are weakened.

5. MULTICRITERIA CONTROL PROBLEMS

In this section we conside problems of the type described in Section 2 except now $g : \mathrm{M} \to \mathrm{E}^p$ is a vector valued function satisfying the Carathéodory conditions. In the finite horizon case this problem has been considered in Cesari and Suryanarayana [7]. In fact, the discussion here is similar to their work.

For such a problem we consider a partial ordering, $<_\Lambda$, defined with respect to a closed convex cone $\Lambda \subseteq E^p$. That is, $x <_\Lambda y$ if and only if $x - y \in \Lambda$. With this ordering we seek an admissible pair $\{x^*, u^*\} \in \Omega$ such that for all $\{x, u\} \in \Omega$

$$(14) \qquad J[x^*, u^*] <_\Lambda J[x, u],$$

and we refer to $\{x^*, u^*\}$ as *a* Λ-Pareto minimum. We require the cone Λ to satisfy the angle property defined as follows.

DEFINITION 1. A closed convex cone $\Lambda \in E^p$ has the *angle property* if there exists ε, $0 < \varepsilon < 1$, and a non-zero vector $a \in \{z \in E^p : \langle z, \lambda \rangle \geq 0$ for all $\lambda \in \Lambda\}$ such that

$$(15) \qquad \Lambda \in \{x \in E^p : \langle a, x \rangle \geq \varepsilon \, |a| \, |x|\},$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in E^p.

The angle property permits us to reduce the above multicriteria problem to scalar valued problem. Specifically we have the following well known result.

PROPOSITION 1. *Let $\Lambda \in E^p$ be a closed convex cone having the angle property, let W be any set, and let $I : W \to E^p$ be a given vector valued function. Then if there exists $c \in E^p$ such that $c <_\Lambda I[x]$ for all $x \in W$ and if $\{I[x] : x \in W\}$ is a closed nonempty set, then any vector $x^* \in W$ satisfying*

$$(16) \qquad \langle a, I[x^*] \rangle = \inf \{\langle a, I[x] \rangle : x \in W\},$$

is a Λ-Pareto minimum of I over W.

To investigate the existence of Λ-Pareto minima we define the orientor field for $(t, x) \in A$ as

$$(17) \qquad \tilde{Q}_\Lambda(t, x) = \{(z^0, z) : z^0 {}_\Lambda\!> g(t, x, u),\ z = f(t, x, u),\ u \in U(t, x)\},$$

and define the Lagrangian $T : [0, +\infty) \times E^n \times E^n \to E^1 \cup \{+\infty\}$ by the formula

$$(18) \qquad T(t, x, z) = \inf \{\langle a, z^0 \rangle : (z^0, z) \in \tilde{Q}_\Lambda(t, x)\},$$

where we assume that $T(t, x, z) = +\infty$ if $(t, x) \notin A$ or if $\tilde{Q}_\Lambda(t, x) = \varnothing$. It is shown in Carlson [5; Thm. 5.2.1.] that under the hypotheses that the sets $\tilde{Q}_\Lambda(t, x)$ are convex and enjoy property (K) the function T is a Lebesgue normal integrand such that $T(t, x, .)$ is convex. This allows us to define the Lagrange problem consisting of minimizing the improper integral

$$I[x] = \int_0^{+\infty} T(t, x(t), \dot{x}(t))\, dt$$

over all arcs $x \in AC_{loc}$. In addition, we show that if x^* is an optimal solution for this scalar Lagrange problem, then there corresponds a control u^* such that the pair $\{x^*, u^*\} \in \Omega$ is a Λ-Pareto minimum. Consequently the existence of a Λ-Pareto minimum can be proved directly through an application of Baum's result [3]. More precisely we have the following result.

THEOREM 4. *Let* A *and* M *be closed and let* $g : M \to E^p$ *and* $f : M \to E^n$ *satisfy the Carathéodory conditons such that the growth conditon* (γ) *holds with g replaced by* $|g|$. *Let* $\Lambda \in E^p$ *be a closed convex cone which has the angle property. Then if there exists* $\alpha \in E^1$ *such that the set*

$$\Omega_\alpha = \left\{\{x, u\} \in \Omega : \int_0^{+\infty} \langle a, g(t, x(t), u(t)) \rangle\, dt \leq \alpha\right\} \neq \varnothing ,$$

the sets $Q_\Lambda(t, x)$ *are closed convex and enjoy property* (K), *and if there exists* $\psi \in L^1([0, +\infty); E^p)$ *such that* $g(t, x, u)_\Lambda > \psi(t)$ *a.e. in* $t \geq 0$, $(t, x, u) \in M$ *the vector valued infinite horizon optimal control problem has a* Λ-*Pareto minimum.*

In addition to the above result, extensions to the Λ-Pareto case are also considered for the weaker notions of optimality when the components of the cost criteria are possibly divergent. For the details see Carlson [5; 5.2].

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REFERENCES

[1] Z. ARSTEIN (1980) – *Discrete and Continuous Bang-Bang and Facial Spaces or : Look for the Extreme Points*, « SIAM Review », *22*, 172-185.

[2] E.J. BALDER (1983) – *An Existence Result for Optimal Economic Growth*, « J. Math. Anal. App. », *95*, 195-213.

[3] R.F. BAUM (1976) – *Existence Theorems for Lagrange Control Problems With Unbounded Time Domain*, « J. Opt. Theory and App. », *19*, 89-116.

[4] D.A. CARLSON – *The Existence of Optimal Solutions for Infinite Horizon Optimal Control Problems*, « Rend. Accad. Nazionale Lincei » (to appear).

[5] D.A. CARLSON (1983) – *On the Existence of Optimal Solutions for Infinite Horizon Optimal Control Problems*, Ph. D. Dissertation, University of Delaware.

[6] L. CESARI (1983) – *Optimization-Theory and Applications*, « Applications of Math. », *17*, Springer-Verlag, New York.

[7] L. CESARI and M.B. SURYANARAYANA (1978) – *Existence Theorems for Pareto Optimization; Multivalued and Banach Space Valued Functionals*, « Trans. Amer. Math. Soc. », *244*, 37-65.

[8] R.T. ROCKAFELLAR (1977) – *Integral Functionals, Normal Integrands, and Measurable Selections, in Nonlinear Operators and The Calculus of Variations* (J. P. Gossez *et al.*, Eds.), Lec. Notes in Math. No. 543, Springer-Verlag, New York.