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RENDICONTI

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *Comparison Theorems for Temperatures in non-cylindrical Domains.* Nota (*) di EUGENE B. FABES (**), NICOLA GAROFALO (***) e SANDRO SALSA (**), presentata dal Socio L. AMERIO.

RIASSUNTO. — In questa Nota gli autori presentano alcuni risultati riguardanti il comportamento alla frontiera di domini non cilindrici delle soluzioni positive dell'equazione del calore. Una conseguenza è che due soluzioni positive qualunque, che si annullano su una parte della frontiera laterale, tendono a zero con lo stesso ordine.

INTRODUCTION

Decisive progress has recently been made in the analysis of the boundary behaviour of positive solutions to uniformly elliptic equations, see [CFMS] and [B] and the references cited there.

Among the results achieved, one stands out for its intimate connection to a fundamental property of harmonic measure (the so-called doubling condition), and the potential theory involved. This result is known as comparison theorem. Roughly speaking, it states that there exists only one rate of convergence to zero for all positive solutions to an elliptic equation that vanish on a portion of the boundary.

As was pointed out by Kemper $[K]_2$, simple examples reveal that a verbatim parabolic imitation of the elliptic result cannot be carried out. This is due

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essentially to the evolutive nature of the phenomenon involved which is reflected in a time-lag in the formulation of the Harnack Principle.

In [FGS] we have proved various comparison results for solutions to divergence type parabolic equations in Lipschitz cylinders, and given applications to non-tangential convergence. In [G] similar results have been obtained for solutions to non-divergence parabolic equations.

In this paper we take up the study started in [FGS] and extend the results that appeared there to the Lip (1,1/2) domains in $\mathbf{R}^{n+1}$ introduced by Kemper $[K]_1$. For the sake of avoiding technical complications we only look at solutions of the heat diffusion operator $H = \Delta - D_t$. However, we wish to emphasize that most of the results presented here may be extended to divergence type parabolic equations, in the spirit of [FGS]. Of course, distinction has to be made for those results, like Theorem 4 below, whose proof relies on techniques that are particular to the constant coefficients case.

This paper is divided into two parts. We have collected the statements of the results in Section 1, postponing to Section 2 the proofs. Theorem 1 is an elliptic type interior Harnack Principle for the special class of the positive solutions of $Hu = 0$ vanishing on the lateral boundary. Theorem 2 gives a global comparison result for every two such solutions. Theorem 3 is the above mentioned local comparison theorem in its parabolic adaptation. A consequence of it is that any two positive solutions of $Hu = 0$, which vanish on a portion of the lateral boundary, go to zero at the same rate.

We have followed in this paper a different order from [FGS], where we first proved the local comparison theorem, and then used it to infer the global one (respectively, Theorems 3 and 2 in this paper). Here we give a proof of the global comparison theorem which is somewhat independent of the local result, and solely relies on the relation between caloric measure and Green's function (see lemma 1 below).

Theorem 4 concludes the results. It is a strengthened version of Harnack Principle at the boundary, and in fact equivalent to the above cited doubling condition for the caloric measure. This is emphasized in the Remark following the theorem.

§ 1. The results

We introduce our basic domains and set up the notation. Ω will represent a bounded domain in $\mathbf{R}^{n+1}$ whose boundary, $\partial\Omega$, is given by $\partial\Omega = B_0 \cup B_T \cup S$, where for every $\tau \in \mathbf{R}$, we indicate with B_τ the hyperplane $\{(x, t) \in \mathbf{R}^{n+1} \mid t = \tau\}$. S is the lateral boundary, and is locally described as the graph of a function satisfying the condition L 2 in $[K]_1$. This means that for every $(Q, s) \in S$ there exists a ball in $\mathbf{R}^{n+1}$, B_{r_0}, of radius r_0 and centered at (Q, s), and a coordinate system of $\mathbf{R}^{n+1}$, $(x, t) = (x', x_n, t)$, in which

$$B_{r_0} \cap S = B_{r_0} \cap \{(x, t) \mid x_n = f(x', t)\}$$

and

$$B_{r_0} \cap \Omega = B_{r_0} \cap \{(x, t) \mid x_n > f(x', t)\} .$$

f is a function globally defined in (x', t) and satisfying

$$|f(x', t) - f(x'_0, t_0)| \leq M(|x' - x'_0| + |t - t_0|^{1/2}) . \tag{1.1}$$

In particular, we exclude the possibility that S contains points (Q, s) around which it is flat, i.e., described by $t = s$. The numbers' r_0 and M, which are assumed independent of $(Q, s) \in S$, are said to determine the Lip (1,1/2) *character of* Ω.

In order to avoid the occurrence of pathologies (see the Remark after Theorem 1), we require each intersection $\Omega \cap B_\tau$, $0 < \tau < T$, to be a simply connected n-dimensional domain.

By a criterion of Petrowski, a domain Ω as described above is a regular domain for the heat equation $Hu = \Delta u - D_t u$, its parabolic boundary being given by $\partial_p \Omega = B_0 \cup S$.

For $(Q, s) \in S$ and $r > 0$ small enough we define

$$\Psi_r(Q, s) = \{(x, t) \in \Omega \mid |x' - Q'| < r, |x_n - Q_n| < rd, |t - s| < r^2\},$$
$$\Delta_r(Q, s) = S \cap \overline{\Psi}_r(Q, s),$$

where $d < 2M$ is a fixed constant. Moreover, we set

$$\overline{A}_r(Q, s) = (Q', Q_n + rd, s + (1 + \mu) r^2),$$
$$\underline{A}_r(Q, s) = (Q', Q_n + rd, s - (1 + \mu) r^2),$$

where $\mu \in (0, 1)$ is chosen (depending on the Lip (1,1/2) character of Ω) so that both $\overline{A}_r(Q, s)$ and $\underline{A}_r(Q, s)$ belong to Ω for small r. We will have occasion to say that a certain constant depends on diam Ω. We define this quantity as $\sup_{0 \leq \tau \leq T} (\operatorname{diam} \Omega \cap B_\tau)$.

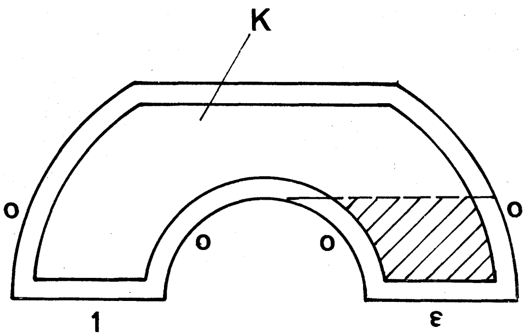
We are now ready to state the first result of this section.

THEOREM 1. (Interior backward Harnack inequality). *Let u be a positive solution of $Hu = 0$ in Ω vanishing continuously on S. For any compact $K \subset \Omega$ there exists a constant C depending only on n, r_0, M, diam Ω, dist (K, S) and dist (K, B_0), such that*

$$\max_K u \leq C \min_K u . \tag{1.2}$$

Remark. It has been observed in [FGS] (see the Remark following Theorem 1.3) that for Theorem 1 to hold one must keep the lateral boundary S at temperature zero. As the example in the picture below shows, the simpleconnectedness of $\Omega \cap B_\tau$ for each $\tau \in (0, T)$ is also necessary.

If u is the positive solution in Ω having boundary values prescribed as in the figure ($u|_S \equiv 0$), we cannot expect (1.2) to hold for K as in the picture with C independent of ε. This can be seen by applying the maximum principle to the shaded part of K below the dotted line.



Remark. While writing this paper we discovered that Jones and T*u* [JT] have proved a result, Lemma 2.1, similar to Theorem 1 above, but for the heat equation in $\mathbf{R}^2$. Their proof is different from ours, but the reasons that led them to formulate an improved Harnack inequality seem to be very much related to those, mentioned in the introduction of [FGS], which led us to conjecture the validity of a backward Harnack Principle, and further improvements of it, like Theorem 4 below.

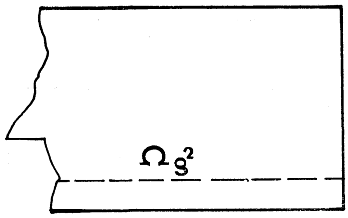
Henceforth we will indicate with Ω_τ the set $\Omega \cap \{(x, t) \mid t > \tau\}$, $\tau \in \mathbf{R}$.

THEOREM 2. (Global comparison Theorem). *Let u, v be two positive solutions of $Hu = 0$ in Ω vanishing continuously on* S, *and let (X_0, T_0) be a fixed point in Ω. If $\delta > 0$ there exists a positive constant* $C = C(n, r_0, M, \delta,$ diam $\Omega)$ *such that*

$$\frac{u(x, t)}{v(x, t)} \leq C \frac{u(X_0, T_0)}{v(X_0, T_0)} \tag{1.3}$$

for all $(x, t) \in \Omega_{\delta^2}$.

Remark. Theorem 2 actually holds for more general Lip (1,1/2) domains than those we have. For instance, in a domain like the one pictured below in which the lateral boundary is allowed to contain flat parts we can adapt the proof of Theorem 2 to obtain (1.3).



THEOREM 3. (Local comparison Theorem). *Let* $(Q, s) \in S$ *and* u, v *be two positive solutions of* $Hu = 0$ *in* $\Psi_{2r}(Q, s)$ *continuously vanishing on* $\Delta_{2r}(Q, s)$. *Then there exists a positive constant* $C = C(n, r_0, M)$, *such that when* r *is sufficiently small we have*

$$\frac{u(x, t)}{v(x, t)} \leq C \frac{u(\bar{A}_r(Q, s))}{v(\underline{A}_r(Q, s))} \tag{1.4}$$

for all $(x, t) \in \Psi_{r/8}(Q, s)$.

For solutions vanishing on S (e.g. the Green's function for H and Ω) Theorem 3 can be improved by means of the following Harnack inequality at the boundary.

THEOREM 4. *Let* u *be a positive solution of* $Hu = 0$ *vanishing continuously on* S, *and let* $\delta > 0$. *There exists a positive constant* $C = C(n, m, r_0, \delta, \text{diam } D)$ *such that for any* $(Q, s) \in S$, *with* $s > \delta^2$, *and* r *sufficiently small*

$$\frac{1}{C} \leq \frac{u(\bar{A}_r(Q, s))}{u(\underline{A}_r(Q, s))} \leq C. \tag{1.5}$$

Remark. If (X_0, T_0) is a suitably fixed point in Ω, and $v(\zeta, \tau) = G(X_0, T_0, \zeta, \tau)$, then the adjoint version of (1.5) for v, together with (2.2) of Lemma 1 in Section 2, can be used to obtain the following doubling condition for caloric measure: for $(Q, s) \in S$ there exists C (independent of r) such that

$$\omega^{(X_0, T_0)}(\Delta_{2r}(Q, s)) \leq C\omega^{(X_0, T_0)}(\Delta_r(C, s)). \tag{1.6}$$

§ 2. THE PROOFS

Proof of Theorem 1. Since $u \in C(K)$, let $(x_0, t_0) \in K$ be such that $u(x_0, t_0) = \min_K u$. Let $\delta = \min(\text{dist}(K, S), \sqrt{\text{dist}(K, B_0)})$, and look at $\Omega_{\Omega^2/2} \supset K$. For points $(Q, \delta^2/2) \in S$ consider the box $\Psi_\eta\left(Q, \frac{\delta^2}{2}\right)$ where $\eta = \frac{\delta}{\sqrt{4(1+\mu)}}$.

It is clear that $\Psi_\eta\left(Q, \frac{\delta^2}{2}\right) \subset \Omega^2/2 \diagup K$, and moreover $\frac{\delta^2}{2} + (1+\mu)\eta^2 = \frac{3\,\delta^2}{4}$. By Lemma 1.3 in [K]$_1$, and Harnack Principle for the heat equation [M], we infer the existence of a constant $C = C(n, M, r_0, \delta, \operatorname{diam}\Omega)$ such that

$$\max_{(x, \frac{\delta^2}{2}) \in \Omega} u\left(x, \frac{\delta^2}{2}\right) \leq C\, u(x_0, t_0).$$

On the other hand, since $u = 0$ on S, the maximum principle implies

$$\max_{\overline{\Omega}^2_{\delta/2}} u \leq \max_{(x, \delta^2/2) \in \Omega} u\left(x, \frac{\delta^2}{2}\right).$$

$\max_K u \leq \max_{\overline{\Omega}^2_{\delta/2}} u$, (2.2) and (2.1) conclude the proof.

Q.E.D.

The next result makes explicit the relation between Green's function and caloric measure. Before stating it we need to recall a few notions. Let

$$\Gamma(x, t; y, s) = \begin{cases} \left(4\pi(t-s)\right)^{-\frac{n}{2}} \exp\left[-\frac{|x-y|^2}{4(t-s)}\right], & t > s \\ 0 & t \leq s \end{cases}$$

be the Gauss-Wierstrass kernel in $\mathbf{R}^{n+1} \diagdown \{(y, s)\}$.

The Green's function $G(x, t; y, s)$ for Ω with pole at (y, s) is defined by

$$G(x, t; y, s) = \Gamma(x, t; y, s) - V(x, t; y, s), \qquad \text{where} \quad V(\cdot, \cdot; y, s)$$

is the solution of the problem

$$\begin{cases} Hu = 0 & \text{in } \Omega \\ u|_{\partial_p \Omega} = \Gamma(\cdot, \cdot; y, s). \end{cases}$$

We also have an adjoint Green's function, corresponding to the operator $H^* = \Delta + D_t$, given by $G^*(y, s; x, t) = G(x, t; y, s)$.

For E a Borel subset of $\partial_p\Omega$ the caloric measure $\omega_\Omega^{(x,t)}(E)$, evaluated at $(x, t) \in \Omega$, is defined as the value at (x, t) of the solution to the problem: $Hu = 0$

in Ω, $u|_{\partial_p\Omega} = \chi_E$, χ_E being the characteristic function of E, in the Perron-Wiener-Brelot sense.

In what follows we will write $\omega^{(x,t)}$ for $\omega_\Omega{}^{(x,t)}$, and use the subscript only if the caloric measure relative to a set different from Ω is involved. Also, if $(y_0, s_0) \in \Omega$, and for r small enough, $B_r(y_0, s_0)$ denotes the ball in $\mathbf{R}^{n+1}$ with centre at (y_0, s_0) and radius r, we define $\Delta_r(y_0, s_0) = B_r(y_0, s_0) \cap B_{s_0} \cap \Omega$. In this situation we set $\bar{A}_r(y_0, s_0) = (y_0, s_0 + r^2)$, $\underline{A}_r = (y_0, s_0 - r^2)$.

LEMMA 1. *Let* $(y_0, s_0) \in \Omega$. *There exists a positive constant* $C = C(n, r_0, M)$ *such that for r sufficiently small and* $(x, t) \in \Omega_{s+4r^2}$

$$(2.1) \qquad \frac{1}{C} r^n G(x, t; \bar{A}_r(y_0, s_0)) \le \omega_{\Omega s_0}^{(x,t)}(\Delta_r(y_0, s_0)) \le C r^n G(x, t; \underline{A}_r(y_0, s_0)).$$

If $(Q, s) \in S$ *a similar result holds, specifically*

$$(2.2) \qquad \frac{1}{C} r^n G(s, t; \bar{A}_r(Q, s)) \le \omega^{(x,t)}(\Delta_r(Q, S)) \le C r^n G(x, t; \bar{A}_r(Q, s))$$

for all $(x, t) \in \Omega_{s+4r^2}$.

Proof. For a point $(y_0, s_0) \in \Omega$ and r small we have

$$(2.3) \qquad \omega_{\Omega s_0}^{(x,t)}(\Delta_r(y_0, s_0)) = \int_{B_{s_0} \cap \Omega} G_{\Omega s_0}(x, t; \zeta, s_0)\, \omega_{\Omega s_0}^{(\zeta, s_0)}(\Delta_r(y_0, s_0))\, d\zeta$$

$$= \int_{\Delta_r(y_0, s_0)} G_{\Omega s_0}(x, t; \zeta, s_0)\, d\zeta,$$

Where $G_{\Omega_{s_0}}$ indicates the Green's function for the domain Ω_{s_0}.

Now using Harnack inequality in the adjoint variables of $G_{\Omega_{s_0}}(x, t; \zeta, s_0)$ in 2.3 (or Lemma 1.3 in $[K]_1$ in its adjoint version if $\Delta_r(y_0, s_0)$ touches the boundary S), and noticing that $G(x, t; .,.,) |_{\Omega_{s_0}} \equiv G_{\Omega_{s_0}}(x, t; .,.)$, we get the right hand side of (2.1). The maximum principle gives the left hand side.

We now examine the case of $(Q, s) \in S$. Choose a $\phi \in C_0^\infty(\mathbf{R}^{n+1})$ such that $\phi \equiv 1$ in $\Psi_r(Q, s)$, and $\phi \equiv 0$ outside $\Psi_{(1+\mu/2)r}(Q, s)$. For $(x, t) \in \Omega_{s+4r^2}$ we have

$$(2.4) \qquad \int_{\mathbf{R}^{n+1}} H\phi(y, s)\, \Gamma(x, t; y, s)\, dy\, ds = -\phi(x, t) = 0.$$

Now we use the known identity

$$(2.5) \qquad G(x,t;y,s) = \Gamma(x,t;y,s) - \int_{\partial_p \Omega} \Gamma(Q,\sigma;y,s)\, d\omega^{(x,t)}(Q,\sigma)$$

which, for fixed $(x,t) \in \Omega$, is valid for any $(y,s) \in \mathbf{R}^{n+1} \setminus \{(x,t)\}$. By (2.4), (2.5), and Fubini's theorem we get

$$(2.6) \qquad \int_{\mathbf{R}^{n+1}} H\phi(y,s)\, G(x,t;y,s)\, dy\, ds =$$

$$= \int_{\mathbf{R}^{n+1}} H\phi(y,s)\{\Gamma(x,t;y,s) - \int_{\delta_p \Omega} \Gamma(Q,\sigma;y,s)\, d\omega^{(x,t)}(Q,\sigma)\}\, dy\, ds =$$

$$= - \int_{\mathbf{R}^{n+1}} H\phi(y,s) \int_{\partial_p \Omega} \Gamma(Q,\sigma;y,s)\, d\omega^{(x,t)}(Q,s)\, dy\, ds$$

$$= - \int_{\partial_p \Omega} \Big(\int_{\mathbf{R}^{n+1}} H\phi(y,s)\, \Gamma(Q,\sigma;y,s)\, dy\, ds \Big)\, d\omega^{(x,t)}(Q,\sigma)$$

$$= \int_{\partial_p \Omega} \phi(Q,\sigma)\, d\omega^{(x,t)}(Q,\sigma) \geq \omega^{(x,t)}(\Delta_r(Q,s)).$$

Now noticing that there exists a $C = C(n)$ such that $|H\phi| \leq Cr^{-2}$, and using the adjoint version of Lemma 1.3 in $[K]_1$ we get

$$(2.7) \qquad \omega^{(x,t)}(\Delta_r(Q,s)) \leq Cr^{-2} \int_{\Psi_{(1+\frac{m}{2})r}(Q,s)} G(x,t;y,s)\, dy\, ds$$

$$\leq Cr^n\, G(x,t;\underline{A}_r(Q,s))$$

for every $(x,t) \in \Omega_{s+4r^2}$. To get the left side of (2.2) we set

$$N_r(x,t) = \left\{ (y,s) \in \Omega \mid |y - x| < \frac{r}{4}, 0 < s - t < \frac{r^2}{16} \right\}.$$

Observe that if $(x,t) \in \partial N_r(\bar{A}_r(Q,s)) \setminus B_{s+(1+\mu)r^2}$ by Lemma 1.1 in $[K]_1$ we have

$$(2.8) \qquad \omega^{(x,t)}(\Delta_r(Q,s)) \geq C$$

with $C = C(n, r_0, M)$. On the other hand since for such points $r^n G(x,t;\bar{A}_r(Q,s)) \leq r^n \Gamma(x,t;\bar{A}_r(Q,s)) \leq C$, the left side of (2.2) follows by an application of the maximum principle to $\Omega_{s+(1+\mu)r^2} \setminus N_r(\bar{A}_r(Q, s))$.

Q.E.D.

In what follows for $\eta, r > 0$ we indicate $D_\eta = \Omega \cap B_\eta$, $D_{\eta,r} = D_\eta \cup \cap \{(x, t) \mid \text{dist}((x, t), S) > r\}$.

COROLLARY 1. *There exists a positive constant* $C = C(n, r_0, M, \eta, r^*,$ diam $D_\eta)$ *such that for any* $(x, t) \in \Omega_{\eta+\epsilon r^{*2}}$.

$$\omega_{\Omega_\eta}^{(x,t)}(D_{\eta,r^*}) \geq C\, \omega_{\Omega_\eta}^{(x,t)}(D_\eta), \tag{2.9}$$

where r^* *is chosen suitably small.*

Proof. Cover $D_\eta \setminus D_{\eta,r^*}$ by a finite number of boxes $\Delta_{4r}^*(x_j, \eta)$, $j = 1, \ldots, Nr^*$, where dist $((x_j, \eta), s) = 2r^*$. Let $\bar{A}_j = (x_j, \eta + r^{*2})$, $\underline{A}_j = (x_j, \eta - r^{*2})$. By the adjoint version of Theorem 1 we infer the existence of a constant $C = C(n, r_0, M, \eta, r^*,$ diam $\Omega)$, such that for any $(x, t) \in \Omega_{\eta+6r^{*2}}$

$$G(x, t; \bar{A}_j) \geq C\, G(x, t; \underline{A}_j), \qquad j = 1, \ldots, N_r^*. \tag{2.10}$$

By Lemma 2 we now deduce that

$$\omega_{\Omega_\eta}^{(x,t)}(\Delta_{4r^*}(x_j, \eta)) \leq C\, \omega_{\Omega_\eta}^{(x,t)}(\Delta_r^*(x_j, \eta)) \tag{2.11}$$

for $j = 1, \ldots, N_{r^*}$, and each $(x, t) \in \Omega_{\eta+6r^{*2}}$.

(2.9) is now an easy consequence of (2.11).

Q.E.D.

Proof of Theorem 2. We may suppose $\delta^2 < T_0$. By Harnack principle and Lemma 1.3 in $[K]_1$ we infer the existence of a constant $C = C(n, r_0, M, \delta,$ diam $\Omega)$ such that for $(x, t) \in \Omega_{\delta^2}$.

$$u(x, t) \leq C\, u(X_0, T_0). \tag{2.12}$$

The maximum principle gives

$$u(x, t) \leq C\, u(X_0, T_0)\, \omega_{\Omega_{\delta^2}}^{(x,t)}(D_{\delta^2}) \tag{2.13}$$

for each $(x, t) \in \Omega_{\delta^2}$. Again by Harnack principle we get for all $(x, t) \in D_{\delta^2, r^*}$

$$v(x, t) \geq C\, v\left(X_0, \frac{\delta^2}{2}\right), \tag{2.14}$$

where r^* is fixed suitably small depending on δ, and C depends on n, r_0, M, δ, diam Ω. The maximum principle implies

$$v(x, t) \geq C\, v\left(X_0, \frac{\delta^2}{2}\right) \omega_{\Omega_{\delta^2}}^{(x,t)} (D_{\delta^2, r^*}) \tag{2.15}$$

for all $(x, t) \in \Omega_{\delta^2}$. By Theorem 1 we get

$$v\left(X_0, \frac{\delta^2}{2}\right) \geq C\, v\, X_0, T_0), \tag{2.16}$$

thus (1.3) follows by (2.13), (2.15), (2.16), and (2.9).

Proof of Theorem 3. Pick $\phi \in C^\infty(R^{n+1})$, $\phi \equiv 1$ in $\Omega \setminus \Psi_{r/2}(Q, s)$, $\phi \equiv 0$ in $\Psi_{r/4}(Q, s)$, and let G_r, ω_r be respectively the Green's function and the caloric measure for $\Psi_r(Q, s)$. We set $\alpha_r = \partial_p \Psi_r(Q, s) \setminus S$ and $\beta_r = \partial_p \Psi_r(Q, s) \cap \{(x, t) \mid x_n = Q_n + rd\}$. As in the proof of Lemma 1 we get for $(x, t) \in \Psi_{r/8}(Q, s)$

$$\begin{aligned} \omega_r^{(x,t)}(\alpha_r) &\leq \int_{\partial_p \Psi_r(Q,s)} \phi(\bar{Q}, \bar{s})\, d\omega_r^{(x,t)}(\bar{Q}, \bar{s}) = \\ &= \int_{R^{n+1}} H\phi(y, s)\, G_r(x, t; y, s)\, dy\, ds \leq \\ &\leq Cr^{-2} \int_{\Psi_{\frac{r}{2}}(Q,s)} G_r(x, t; y, s), dy\, ds \leq Cr^n G_r(x, t; \underline{A}_{r/2}) \leq \\ &\leq C\omega_r^{(x,t)}(\beta_r). \end{aligned} \tag{2.17}$$

In the last two inequalities we have used Lemma 1.3 in $[K]_1$ and Lemma 1 for the set $\Psi_r(Q, s)$. Now let u, v be as in the statement of Theorem 2. We have for all $(x, t) \in \Psi_{r/8}(Q, s)$:

$$u(x, t) \leq Cu(\bar{A}_r(Q, s))\, \omega_r^{(x,t)}(\alpha_r), \tag{2.18}$$

by Lemma 1.3 in $[K]_1$ and maximum principle,

$$v(x, t) \geq C\, v(\underline{A}_r(Q, s))\, \omega_r^{(x,t)}(\beta_r), \tag{2.19}$$

by Harnack and maximum principles. Putting (2.18), (2.19), and (2.17) together we get (1.4). Q.E.D.

Proof of Theorem 4. Since $u=0$ on S we can write

$$(2.20)\qquad u(\bar{A}_r(Q,s)) = \int_{D_{\frac{\delta^2}{2}}} u\left(\xi, \frac{\delta^2}{2}\right) G\left(\bar{A}_r(Q,s); \xi, \frac{\delta^2}{2}\right) d\xi$$

$$u\,\underline{A}_r(Q,s)) = \int_{D_{\frac{\delta^2}{2}}} u\left(\xi, \frac{\delta^2}{2}\right) G\left(\underline{A}_r(Q,s); \xi, \frac{\delta^2}{2}\right) d\xi\,.$$

Now consider the two functions $v_1(\xi,\tau) = G(\bar{A}_r(Q,s); \xi,\tau)$, $v_2(\xi,\tau) = G(\underline{A}_r(Q,s); \xi,\tau) \cdot v_1, v_2$ are two positive solutions of $H^* v = 0$ in $\Omega \setminus \Omega_{\frac{3}{4}\delta^2}$ vanishing on the lateral part of the adjoint parabolic boundary. By the adjoint version of Theorem 2 we get

$$(2.21)\qquad \sup_{(\xi,\frac{\delta^2}{2}) \in D_{\frac{\delta^2}{2}}} \frac{G(\bar{A}_r(Q,s); \xi, \frac{\delta^2}{2})}{G(\underline{A}_r(Q,s); \xi, \frac{\delta^2}{2})} \leq C \frac{G(\bar{A}_r(Q,s); X^*, T^*)}{G(\underline{A}_r(Q,s); X^*, T^*)}$$

for a certain constant $C = C(n, m, r_0, \delta, \operatorname{diam} D)$, where (X^*, T^*) is a suitably fixed point in $\Omega \setminus \Omega_{\delta/4}$. Now using the adjoint version of Lemma 2.2 in [W] we infer the existence of a constant C such that

$$(2.22)\qquad \frac{G(\bar{A}_r(Q,s_r); X^*, T^*)}{G(\underline{A}_r(Q,s_r); X^*, T^*)} \leq C\,.$$

(2.20), (2.21), and (2.22) imply the right hand side inequality in (1.5). The one on the left is just a consequence of Harnack principle. Q.E.D.

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