Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Nonlinear analysis. New arguments and results. II.

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **77** (1984), n.1-2, p. 13–20.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1984_8_77_1-2_13_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1984.

Analisi matematica. — *Nonlinear analysis. New arguments and results.* Nota II (*) del Socio straniero LAMBERTO CESARI.

We continue here the discussion in part I, and we state and prove further sufficient conditions for the existence of a solution to nonselfadjoint problems.

6. ANOTHER COROLLARY

We state and prove here, as a further corollary of (4.i), a Landesman-Lazer type theorem for nonselfadjoint problems where again the Shaw condition is not required. For the sake of simplicity we present it here in the situation depicted by Corollary (4.ii) and for real valued functions. Thus we assume that $X = Y$ is a space of bounded functions in G, $G \subset \mathbf{R}^\nu$, $\nu \geq 1$, with values in $\mathbf{R}$.

We further take $Nx = f(t) + g(t, x(t))$, $t \in G$, $x \in X$, with $f : G \to \mathbf{R}$, $g : G \times \mathbf{R} \to \mathbf{R}$, f and g bounded. The constants L, d, h are now such that $\| Hy \|_\infty \leq L \| y \|_\infty$ for all $y \in Y_1$, and $\| SQy \|_\infty \leq d \| y \|_\infty$, $\| (I - Q) y \|_\infty \leq h \| y \|_\infty$ for all $y \in Y$, and as in no. 3 we assume that σ is the identity map, M is non singular, and $SQ\sigma$ is the identity map.

(6.i) COROLLARY. *Let* $g : G \times \mathbf{R} \to \mathbf{R}$ *be a function satisfying*

$$xg(t, x) \geq 0 \, , \quad |g(t, x)| \leq C \, , \quad |g(t, x) - g(t, y)| \leq \leq D |x - y| \, , \; t \in \mathbf{R} \, , x, y \in \mathbf{R} \, . \tag{9}$$

(10) *There are constants* λ, $0 < \lambda < C$, *sufficiently close to* C, *and* $\bar{R} > 0$ *such that* $C \geq g(t, x) \geq \lambda$ *for* $x \geq \bar{R}$, $t \in \mathbf{R}$, *and* $-C \leq g(t, x) \leq -\lambda$ *for* $x \leq -\bar{R}$, $t \in \mathbf{R}$.

Let T_1 *be compact,* Y_0 *finite dimensional, and let us assume that for the constant* d *of no.* 4 *we have* $d < 2(1 + DL\,h)^{-1}$. *Then there are numbers* c, R_0, $r > 0$, $r' \geq 0$ *such that if* $\| f \|_\infty \leq c$, *problem* $Ex = f(t) + g(t, x)$ *has at least one solution* $x = x_{01} + x_{02} + x_1$, $x_{01} \in X_{01}$, $x_{02} \in X_{02}$, $x_1 \in X_1$, $\| x_{01} \|_\infty \leq R_0$, $\| x_{02} \|_\infty \leq r'$, $\| x_1 \|_\infty \leq r$. *If* $p = q$ *then* $r' = 0$ *and* $x_{02} = 0$; *if* $p > q$, *then* $r' > 0$ *and the problem has at least one solution* x *for every* $x_{02} \in X_{02}$, $\| x_{02} \|_\infty \leq r'$.

(*) Presentata nella seduta del 15 giugno 1984.

Proof. We take for σ the identity map, and we define $S : Y_0 \to X_{01}$ as in no. 3. Let $S_{01} = [x_{01} \in X_{01}, \| x_{01} \|_\infty \leq R_0]$, $S_1 = [x_1 \in X_1, \| x_1 \|_\infty \leq r]$, and let $x_{02} \in X_{02}$, $\| x_{02} \| \leq r'$ (thus, $r' = 0$, $x_{02} = 0$ if $p = q$). We have only to show that we can determine $c, r, R_0, \rho > 0$ and $r' \geq 0$ such that

$$(11) \qquad | x - kg(t, x) | \leq \rho R_0 \quad \text{for all} \quad | x | \leq R_0, \ t \in G,$$

$$(12) \qquad Lh(c + C) \leq r,$$

$$(13) \qquad \rho d < 1,$$

$$(14) \qquad kd\, c + kd\, D(r + r') \leq (1 - \rho d) R_0 .$$

First we note that $d < 2(1 + DLh)^{-1}$ implies $1/2 < 1/d - DL\,h/2$, and then for $0 < \lambda < C$, λ sufficiently close to C, and for $c > 0$ sufficiently small, we also have

$$(15) \qquad 1 - \lambda/2\,C < 1/d - (1 + DLh)(c/2\,C) - DL\,h/2 .$$

Let $k > 0$ be so chosen that $Ck \geq \bar{R}$, and take $R_0 = 2\,Ck$. Hence, $R_0 > > Ck \geq \bar{R}$. Now

$$| x - kg(t, x) | \leq kC \quad \text{for} \quad 0 \leq x \leq kC \text{ and for } -kC \leq x \leq 0;$$

$$| x - kg(t, x) | \leq R_0 - k\lambda \quad \text{for} \quad kC \leq x \leq R_0 \quad \text{and for } -R_0 \leq x \leq -kC,$$

and relation (11) certainly holds provided $kC \leq \rho R_0$ and $R_0 - k\lambda \leq \rho R_0$, or

$$\rho \geq 1/2 \qquad \text{and} \quad \rho \geq 1 - (\lambda/2\,C) .$$

The last requirement implies the previous one. To satisfy (12) we just take $r = Lh(c + C)$. Then, for $r' = 0$, relation (14) becomes

$$kd\, c + kd\, D \cdot Lh(c + C) \leq (1 - \rho d) R_0 = (1 - \rho d)(2\,Ck) . \quad \text{or}$$

$$\rho \leq 1/d - (1 + DLh)(c/2\,C) - DLh/2 .$$

By (15) we see that it is possible to satisfy the requirements on ρ by taking

$$1 - (\lambda/2\,C) < \rho < 1/d - (1 + DLh))(c/2\,C) - DLh/2 .$$

Note that necessarily $\rho < 1/d$, or $\rho d < 1$, and (13) also is satisfied. Now we have determined c, R_0, r, ρ so that all relations (11-14) are satisfied, in particular relation (14) is satisfied with $r' = 0$ and the $<$ sign. Thus, when $p > q$ we can also determine $r' > 0$ sufficiently small so that (14) holds as written.

7. Another sufficient condition

Here we present sufficient conditions of a different type. We shall still make use of the decomposition $X = X_{01} + X_{02} + X_1$ as in no. 3, but we shall require less on the map $S : Y_0 \to X_{01}$. Indeed, we assume that Y_0 and X_{01} have finite dimension q, and that $S : Y_0 \to X_{01}$ is a linear transformation with

$S^{-1}(0) = 0$. Thus, for given orthonormal bases $X_{01} = sp(\phi_1, \ldots, \phi_q)$, $Y_0 = (\omega_1, \ldots, \omega_q)$, S is represented by a non singular $q \times q$ matrix $M^* = [m^*_{ij}, i, j = 1, \ldots, q]$ with $c = M^* d$, $x = \Sigma_1^q c_i \phi_i \in X_{01}$, $y = \Sigma_1^q d_i \omega_i \in Y_0$, $c = \mathrm{col}(c_1, \ldots, c_q)$, $d = \mathrm{col}(d_1, \ldots, d_q)$, $c_i = \Sigma_j m^*_{ij} d_j$, $i = 1, \ldots, q$. Then, there is a constant $M_0 > 0$ such that $\| Sy \| \leq M_0 \| y \|$ for all $y \in Y_0$.

(7.i) THEOREM. *Let* $X = Y = L_2(G)$, *let* P, Q *be the orthogonal projections with* $PX = X_0 = \ker E$, $QY = Y_0 = \ker E^*$, $\infty \geq p \geq q \geq 0$, $p = \dim X_0$, $q = \dim Y_0$, $X = X_0 + X_1$, $Y = Y_0 + Y_1$, *and a further decomposition* $X_0 = X_{01} + X_{02}$, $X_{01} = sp(\phi_1, \ldots, \phi_q)$, $Y_0 = sp(\omega_1, \ldots, \omega_q)$, $q < \infty$. *Let us assume that, for the given problem* $Ex = Nx$, *the map* H *is compact, and there are constants* J_0, $R_0 > 0$, $r \geq LJ_0$, $r' \geq 0$, *such that* (B) $\| Nx \| \leq J_0$ *for all* $x \in X$; *and* (N_0) $(SQNx, x_{01}) \leq 0$ (*or* ≥ 0) *for all* $x = x_{01} + x_{02} + x_1$, $x_{01} \in X_{01}$, $x_{02} \in X_{02}$, $x_1 \in X_1$, $\| x_{01} \| \geq R_0$, $| x_{02} | \leq r'$, $\| x_1 \| \leq r$. *Then the equation* $Ex = Nx$ *has at least one solution* $x \in X$, $x = x_{01} + x_{02} + x_1$, $x_{01} \in X_{01}$, $x_{02} \in X_{02}$, $x_1 \in X_1$, $\| x_{02} \| \leq r'$, $\| x_1 \| \leq r$. *Actually, if* $p = q$, *then* $r' = 0$, $x_{02} = 0$; *if* $p > q$, *then* $r' > 0$, *and the problem has at least one solution for every* $x_{02} \in X_{02}$, $\| x_{02} \| \leq r'$.

This statement is only a modification of the theorem concerning Hilbert spaces proved by Cesari and Kannan ([3b], p. 222) by Schauder's fixed point theorem (see also Cesari ([1e], (34.ii), p. 126) and Cesari and Kannan ([3c], (2.i), p. 752)). In the latter reference a different proof by Kannan and McKenna was also given. For Banach spaces an analogous theorem was proved by Cesari ([1f], Th. 1, p. 46) (see also Cesari ([1e], (37.i), p. 140), and for the case of unbounded non linearity with limited growth (Cesari [1f], Th. 1*, p. 49) again by Schauder's fixed point theorem.

LEMMA. *Let* $g : \mathbf{R} \to \mathbf{R}$ *be a given continuous function with finite limits* $g(+\infty)$, $g(-\infty)$. *Let* G *be a fixed measurable point set in* $\mathbf{R}^\nu$, $\nu \geq 1$, *of finite measure* $| G |$. *Let* $\phi = (\phi_1, \ldots, \phi_q)$ *be a given orthonormal system of functions* $\phi_i \in L_2(G)$ *with the property that every function* $w = b_1 \phi_1 + \cdots + b_q \phi_l$, $t \in G$, $| b | = (b_1^2 + \ldots + b_q^2)^{\frac{1}{2}} = 1$, *is zero in* G *at most in a set of measure zero. For any such* w, *let* $G_+ = [t \in G \mid w(t) > 0]$, $G_- = [t \in G \mid w(t) < 0]$, *so that* $| G_+ | + | G_- | = | G |$. *Let* $M, r > 0$ *be given constants. Then there is some* $R_0 > 0$ *such that for all* $\rho \geq R_0$, *for all* b *with* $| b | = 1$, *and for all functions* z, $F \in L_2(G)$ *with* $\| z \| \leq r$, $\| F \| \leq M$, *we have*

$$\left| \int_{G+} [g(+\infty) - g(\rho w(t) + z(t))] F(t) \, dt \right| < \varepsilon,$$

$$\left| \int_{G-} [g(-\infty) - g(\rho w(t) + z(t))] F(t) \, dt \right| < \varepsilon.$$

In other words, both integrals approach zero as $\rho \to +\infty$ *uniformly with respect to* b, z, F *with* $| b | = 1$, $\| z \| \leq r$, $\| F \| \leq M$.

Proof. Here g is necessarily bounded, say $|g(x)| \leq C$ for all $x \in \mathbf{R}$; hence, $|g(+\infty)|$, $|g(-\infty)| \leq C$, and $|g(+\infty) - g(x)|$, $|g(-\infty) - g(x)| \leq 2C$ for all $x \in \mathbf{R}$. Here G has finite measure, and each function $w(t)$ of the collection $[w(t) = b_1 \phi_1 + \cdots + b_q \phi_q,\ |b| = 1]$, has the property that $w(t) = 0$ at most in a set of measure zero in G (which may depend on w). Then, by [3 c, (10.iv), p. 773], given $\varepsilon > 0$ there is $\gamma = \gamma(\varepsilon) > 0$, independent of $b = (b_1, \ldots, b_q)$, such that the set of points of G where $|w(t)| \leq \gamma$ has measure $\leq \varepsilon$. (We have used (10.iv) of [3c] for $\lambda_0 = 0$ and a collection which need not be the kernel of an operator). Then, for $\gamma = \gamma((12\,CM)^{-2}\varepsilon^2)$ we have

$$|[t \in G \mid |w(t)| = |b_1 \phi_1 + \cdots b_q \phi_q| \leq \gamma,\ |b| = 1]| \leq (12\,CM)^{-2}\varepsilon^2 .$$

Thus, for $A_+ = [t \in G_+ \mid 0 < w(t) \leq \gamma]$, then $|A_+| \leq (12\,CM)^{-2}\varepsilon^2$, and

$$\left| \int_{A_+} [g(+\infty) - g(\rho w(t) + z(t))]\, F(t)\, dt \right| \leq 2C \int_{A_+} |F(t)|\, dt$$
$$\leq 2C\, |A_+|^{1/2}\, \|F\| \leq \varepsilon\,(2\,CM)\,(12\,CM)^{-1} = \varepsilon/6 .$$

Let $K \geq (12\,CMr)\,\varepsilon^{-1}$ and take $B_+ = [t \in G_+ \mid |z(t)| \geq K]$. Then

$$r^2 = \int_G z^2\, dt \geq \int_{B_+} z^2\, dt \geq K^2\, |B_+|, \qquad |B_+| \leq r^2 K^{-2},$$

and

$$\left| \int_{B_+} [g(+\infty) - g(\rho w(t) + z(t))]\, F(t)\, dt \right| \leq 2C \int_{B_+} |F|\, dt$$
$$\leq 2C\, |B_+|^{1/2}\, \|F\| \leq 2\,CM\,(r K^{-1}) \leq \varepsilon/6 .$$

The given function g has finite limits $g(+\infty)$, $g(-\infty)$. Hence, given ε there is $R > 0$ such that

$$|g(+\infty) - g(x)| \leq (6\,M)^{-1}\, |G|^{-1/2}\, \varepsilon \qquad \text{for all } x \geq R,$$

$$|g(-\infty) - g(x)| \leq (6\,M)^{-1}\, |G|^{-1/2}\, \varepsilon \qquad \text{for all } x \leq -R,$$

and R depends on the given function g, on G and M, but not on the specific function F with $\|F\| \leq M$. Now take $R_0 \geq \gamma^{-1}(K + R)$, and $C_+ = G_+ - A_+ - B_+$. Then $|z(t)| \leq K$ and $w(t) \geq \gamma$ for all $t \in C_+$. Hence, for $\rho \geq R_0$ we have

$$\rho w(t) + z(t) \geq R_0\, \gamma - K \geq R \qquad \text{for } t \in C_+ .$$

Consequently,

$$|g(+\infty) - g(\rho w(t) + z(t))| \leq (6\,M)^{-1}\,|G|^{-1/2}\,\varepsilon \qquad \text{for} \quad t \in C_+ ,\, \rho \geq R_0,$$

and

$$\left|\int_{C_+} [g(+\infty) - g(\rho\, w(t) + z(t))]\,F(t)\,dt\right| \leq (6\,M)^{-1}\,|G|^{-1/2}\,\varepsilon \int_{C_+} |F|\,dt$$

$$\leq (6\,M)^{-1}\,|G|^{-1/2}\,(|G|^{1/2}\,M)\,\varepsilon = \varepsilon/6 \qquad \text{for} \quad \rho \geq R_0 .$$

Thus for $\rho \geq R_0$ we have

$$\left|\int_{G_+} [g(+\infty) - g(\rho\, w(t) + z(t))]\,F(t)\right| =$$

$$= \left|\left(\int_{A_+} + \int_{B_+} + \int_{C+}\right)\right| \leq \varepsilon/6 + \varepsilon/6 + \varepsilon/6 = \varepsilon/2 .$$

The same argument holds for G_- and

$$\left|\int_{G} [g(+\infty) - g(\rho\, w(t) + z(t))]\,F(t)\right| = \int_{G_+} + \int_{G_-} \leq$$

$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon \qquad \text{for} \quad \rho \geq R_0 ,$$

and R_0 has been chosen independently of the particular vector b and particular functions z, F with $|b| = 1$, $\|z\| \leq r$, $\|F\| \leq M$. This proves the lemma.

For any element $w \in X_{01}$ with $\|w\| = 1$, or equivalently, $w = b_1 \phi_1 + \cdots + b_q \phi_q$, $|b| = 1$, let us construct the new function $W(t)$, $t \in G$, defined by

$$W(t) = \Sigma_s\, \omega_s(t) \int_G \Sigma_i\, m^*_{is}\, \phi_i(\alpha)\, w(\alpha)\, d\alpha\,, \quad t \in G .$$

Also, as before, let G_+, G_- denote the subsets of G where $w(t) > 0$, $w(t) < 0$ respectively. Note that in the norm of $L_2(G)$ we have $\|w\| = \|b_1 \phi_1 + \ldots + b_q \phi_q\| = |b| = 1$,

$$\left|\int_G \phi_i(\alpha)\, w(\alpha)\, d\alpha\right| \leq \|\phi_i\|\, \|w\| = 1 \;,$$

$$\int_G |\phi_i(\alpha)|\, |w(\alpha)|\, d\alpha \leq \|\phi_i\|\, \|w\| = 1 ,$$

and

$$\| W \| \leq \Sigma_s \| \omega_s \| \left| \int_G \Sigma_i m_{is}^* \phi_i(\alpha) w(\alpha) \, d\alpha \right|$$

$$\leq \Sigma_i \Sigma_s | m_{is}^* | \| \omega_s \| \int_G | \phi_i(\alpha) | | w(\alpha) | \, d\alpha \leq \Sigma_i \Sigma_s | m_{is}^* | = \mu ,$$

a fixed number depending only on S, that is, on the matrix $[m_{is}^*]$.

We consider now the problem $Ex = f(t) + g(x(t))$, $t \in G$, where E is an elliptic uniform differential operator on G with associated boundary conditions, where $f \in L_2(G)$, and $g : \mathbf{R} \to \mathbf{R}$ is a continuous function with finite limits $g(-\infty)$, $g(+\infty)$. We assume for E the main properties: (*a*) $\infty > p \geq q \geq 0$, $p = \dim \ker E$, $q = \dim \ker E^*$, with given decompositions and orthonormal bases $X_0 = \ker E = X_{01} + X_{02}$, $X_{01} = sp(\phi_1, \ldots, \phi_q)$, $X_{02} = (\phi_{q+1}, \ldots, \phi_p)$, $Y_0 = \ker E^* = sp(\omega_1, \ldots, \omega_q)$, and we define $S : Y_0 \to X_{01}$ by an arbitrary nonsingular $q \times q$ matrix $M^* = [m_{is}^*]$. (b) Every element $w \in X_{01}$, $w(t) = b_1 \phi_1 + \cdots + b_q \phi_q$, $|b| = 1$, is zero at most in a set of measure zero in G. As for Theorem (7.1), let $X = Y = L_2(G)$, and let P, Q be the orthogonal projections with $PX = X_0 = \ker E$, $QY = Y_0 = \ker E^*$.

(7.ii) THEOREM *Under the above assumptions, if for every* $w \in X_{01}$, $w(t) = b_1 \phi_1 + \cdots + b_q \phi_q$, $|b| = 1$, *we have*

$$\Delta = \int_G f(t) W(t) \, dt + g(+\infty) \int_{G_+} W(t) \, dt + g(-\infty) \int_{G_-} W(t) \, dt > 0 \ (\text{or } < 0) , \tag{18}$$

then there are numbers $R, r > 0$, $r' \geq 0$ *such that the equation* $Ex = f + g(x)$ *has at least one solution* $x = x_{01} + x_{02} + x_1$, $x_{01} \in X_{01}$, $x_{02} \in X_{02}$, $x_1 \in X_1$, $\| x_{01} \| \leq R$, $\| x_{02} \| \leq r'$, $\| x_1 \| \leq r$. *Actually, if* $p = q$, *then* $r' = 0$, $x_{02} = 0$; *if* $p > q$, *then* $r' > 0$, *and the problem has at least one solution for every* $x_{02} \in X_{02}$, $\| x_{02} \| \leq r'$.

Proof. Here g is bounded, say $|g(x)| \leq C$ for all $x \in \mathbf{R}$, hence $\| Nx \| = \| f + g(x) \| \leq \| f \| + |G|^{1/2} C = J_0$, and assumption (B) of (7.i) is satisfied. To prove that (N_0) also is satisfied, we must find first a suitable expression for $(SQNx, w)$ for $w = \rho w(t) + x_{02}(t) + x_1(t)$, $\rho > 0$, $w(t) = b_1 \phi_1 + \cdots + b_q \phi_q$, $|b| = 1$. Indeed, $QNx = \Sigma_s d_s \omega_s$, $d = \text{col}(d_1, \ldots, d_q)$, $d_s = (Nx, \omega_s)$, $s = 1, \ldots, q$, and

$$(QNx)(t) = \Sigma_s \left(\int_G (Nx)(\alpha) \, \omega_s(\alpha) \, d\alpha \right) \omega_s(t) .$$

Also we have

$$\mathrm{SQN}x = \Sigma_i c_i \phi_i , c = \mathrm{col}(c_1 , \ldots , c_q) , c = \mathrm{M}^* \mathrm{d} , c_i = \Sigma_s m_{is}^* \mathrm{d}_s ,$$

$$(\mathrm{SQN}x)(t) = \Sigma_i \Sigma_s m_{is}^* \mathrm{d}_s \phi_i(t) = \Sigma_i \Sigma_s m_{is}^* (\mathrm{N}x , \omega_s) \phi_i(t) ,$$

$$\begin{aligned}(\mathrm{SQN}x , w) &= \int_G (\mathrm{SQN}x)(t)\, w(t)\, \mathrm{d}t \\ &= \int_G \Sigma_i \Sigma_s m_{is}^* \phi_i(t)\, w(t)\, \mathrm{d}t \int_G (\mathrm{N}x)(\alpha)\, \omega_s(\alpha)\, \mathrm{d}\alpha \\ &= \int_G (\mathrm{N}x)(\alpha)\, \Sigma_s \omega_s(\alpha)\, \mathrm{d}\alpha \int_G \Sigma_i m_{is}^* \phi_i(t)\, w(t)\, \mathrm{d}t .\end{aligned}$$

We have proved that

$$(\mathrm{SQN}x , w) = \int_G f(t)\, \mathrm{W}(t)\, \mathrm{d}t + \int_G g(\rho w(t) + x_{02}(t) + x_1(t))\, \mathrm{W}(t)\, \mathrm{d}t .$$

The lemma shows that this expression approaches Δ as $\rho \to +\infty$ uniformly with respect to b , x_{02} , x_1 with $|b| = 1$, $\| x_{02} \| \leq r'$, $\| x_1 \| \leq r$. Also we note that Δ is a continuous function of w , x_{02} , x_1 so that there is some $\varepsilon > 0$ such that $\Delta \leq -2\varepsilon$, or $\Delta \geq 2\varepsilon$, for all b , x_{02} , x_1 as stated. Let R_0 be such that $|(\mathrm{SQN}x , w) - \Delta| < \varepsilon$ for all $\rho \geq \mathrm{R}_0$ and all b , x_{02} , x_1 as stated. Then $(\mathrm{SQN}x , w) \leq -\varepsilon$ (or $> \varepsilon$) for all $\rho \geq \mathrm{R}_0$, $w \in \mathrm{X}_{01}$, $\| w \| = 1$, and x_{02} , x_1 as stated, and finally

$$((\mathrm{SQN})(x_{01} + x_{02} + x_1) , x_{01}) < 0 , \ (\text{or} > 0)$$

for all $\| x_{01} \| \geq \mathrm{R}_0$, $\| x_{02} \| \leq r'$, $\| x_1 \| \leq r$. We have proved that assumption (N_0) of Theorem (7.i) holds. Theorem (7.ii) is thereby proved.

Remark. For $p = q$, $\phi_i = \omega_i$, $m_{is} = \delta_{is}$, then $\mathrm{W}(t) = w(t)$, and Theorem (7.ii) reduces to the Landesman and Lazer sufficiency condition. A discussion on condition (18) and further considerations will appear later.

Remark. Note that $w = b_1 \phi_1 + \cdots + b_q \phi_q$, $|b| = 1$, so that $\int_G \phi_i w \mathrm{d}t = b_i$, $i = 1 , \ldots , q$, and then W becomes $\mathrm{W} = \Sigma_s \Sigma_i m_{is} b_i \omega_s = \Sigma_s \beta_s \omega_s$ with $\beta_s = \Sigma_i m_{is}^* b_i$, $s = 1 , \ldots , q$.

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