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The existence of optimal solutions for infinite horizon optimal control problems

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Analisi matematica. — *The Existence of Optimal Solutions for Infinite Horizon Optimal Control Problems.* Nota di DEAN A. CARLSON (*), presentata (**) dal Socio straniero L. CESARI.

RIASSUNTO. — Si considerano problemi di controllo ottimo del tipo di Lagrange su un intervallo infinito $(0, +\infty)$. In connessione con esempi di problemi di economia non si assume a priori che il costo funzionale, un integrale improprio, sia finito. Si considerano pertanto vari problemi di ottimizzazione che sono più generali di quelli che sono apparsi recentemente nella letteratura. Si presentano vari risultati circa l'esistenza di soluzioni ottimali. Allo scopo si utilizzano proprietà di convessità e di seminormalità analoghe a quelle studiate da Cesari per problemi su un intervallo finito.

1. INTRODUCTION

We consider the existence of optimal solutions for problems of optimal control defined on the unbounded time interval $[0, +\infty)$. As early as 1940, Cinquini [8] investigated free problems of the Calculus of Variations with a Lagrange-type functional of the form

$$\int_0^{+\infty} f(t, x(t), \dot{x}(t))\, dt .$$

There, as in most subsequent investigations, only those absolutely continuous functions $x = x(t)$, $t \geq 0$, for which the above improper integral is finite were considered admissible. Recent examples arising in economics (see e.g. [7] or [5; 1.3]) demonstrate the need to consider functions for which (1) may diverge thus rendering inadequate the traditional definition of optimality. In order to study such problems, a heirarchy of weaker concepts of optimality has been introduced as we describe below.

The question of existence of optimal solutions with respect to these weaker notions has been investigated in [4], [9], and [11]. There, convexity hypotheses stronger than those utilized when considering the traditional definition of optimality are made (see e.g. [3], [2], and [1]). Here, we present several results where these strong convexity hypotheses are replaced by the now classical convexity and seminormality conditions in accord with the work of Cesari on problems of optimal control with finite horizon [6].

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2. Concepts of Optimality

We consider an optimal control problem consisting of minimizing an integral functional of the form

$$\text{(2)} \qquad J[x, u] = \int_0^{+\infty} g(t, x(t), u(t))\, dt$$

over all pairs $\{x, u\} : [0, +\infty) \to E^{n+m}$ satisfying a control system described by

$$\text{(3)} \qquad \dot{x}(t) = f(t, x(t), u(t)) \quad , \quad \text{a.e. on } [0, +\infty) ;$$

subject to state constraints

$$\text{(4)} \qquad (t, x(t)) \in A , \quad \text{on } [0, +\infty) ;$$

control constraints

$$\text{(5)} \qquad u(t) \in U(t, x(t)) , \quad \text{a.e. on } [0, +\infty);$$

and the fixed initial condition

$$\text{(6)} \qquad x(0) = x_0 .$$

Here $A \subseteq [0, +\infty) \times E^n$ is a closed set and U is a set-valued map from A into E^m with closed graph $M = \{(t, x, u) : (t, x) \in A , u \in U(t, x)\}$. The function $g : M \to E^1$ is a normal integrand in the sense of Rockafellar [10] and $f : M \to E^n$ is a Carathéodory function.

The pairs of functions $\{x, u\}$ satisfying the constraints (3) to (6) will be called admissible if x is locally absolutely continuous, u is Lebesgue measurable, and if the map $t \to g(t, x(t), u(t))$ is locally L-integrable. The class of all admissible pairs is denoted by Ω. We assume that $\Omega \neq \varnothing$, and refer to the problem of optimizing J over Ω as $\mathscr{P}$.

We now define the weaker notions of optimality referred to above.

Definition 1. Let $\Delta(T) = \int_0^T [g(t, x(t), u(t)) - g(t, x^*(t), u^*(t))]\, dt$.

An admissible pair $\{x^*, u^*\}$ is called:

(i) *strongly optimal* if $J[x^*, u^*] < +\infty$ and if for all $\{x, u\} \in \Omega$

$$\lim_{T \to +\infty} \int_0^T g(t, x(t), u(t))\, dt \geq J[x^*, u^*];$$

(ii) *overtaking optimal* if, for all $\{x, u\} \in \Omega$, $\lim_{T \to +\infty} \Delta(T) \geq 0$;

(iii) *catching up optimal* if, for all $\{x, u\} \in \Omega$, $\liminf_{T \to +\infty} \Delta(T) \geq 0$;

(iv) *sporadically catching up optimal* if, for all $\{x, u\} \in \Omega$, $\limsup_{T \to +\infty} \Delta(T) \geq 0$; and

(v) *finitely optimal* if for each $T > 0$ and all $\{x, u\} \in \Omega$, with $x(T) = x^*(T)$, $\Delta(T) \geq 0$.

These definitions are listed in descending order, with the notion of strong optimality coinciding with the usual definition of a true minimum. Examples show that these notions of optimality are independent. For details, see Carlson [5 ; 1.3]. For brevity we consider here solutions which are optimal in he sense of (iii) and (v).

The convexity and seminormality conditions we require are expressed in terms of the orientor field for the system defined on A by

$$\tilde{Q}(t, x) = \{(z^0, z) : z^0 \geq g(t, x, u),\ z = f(t, x, u),\ u \in U(t, x)\}.$$

We assume these sets are convex and satisfy the upper semicontinuity condition property (K) given by

$$\text{(K)} \qquad \tilde{Q}(t, x) = \bigcap_{\delta > 0} \operatorname{cl}[\cup \{\tilde{Q}(t, y) : (t, y) \in A,\ |x - y| < \delta\}],$$

where $|.|$ denotes the usual euclidean metric in E^n.

REMARK 1. The results of [4], [9], and [11] require the sets $\Gamma(t) = \{(x, z^0, z) : (t, x) \in A,\ (z^0, z) \in \tilde{Q}(t, x)\}$ to be convex for $t \geq 0$. Clearly this assumption is stronger than the convexity hypothesis given above.

Finally, we consider the set of admissible trajectories as a subset of the space AC_{loc} of locally absolutely continuous functions equipped with the topology of convergence of initial values and weak L^1-convergence of derivatives on compact subsets of $[0, +\infty)$. The needed weak compactness of the set of admissible trajectories is insured by the use of an extension of the standard growth condition (γ) utilized by Cesari [6; 10.4] and others.

DEFINITION 2. The functions f and g are said to satisfy the growth condition (γ) provided for each $\varepsilon > 0$ and $T > 0$, there exists and $L_T > 0$ and $\psi_{\varepsilon T} \in L^1([0, T])$ such that $g(t, x, u) \leq L_T$ and $|f(t, u, u)| \leq \psi_{\varepsilon T}(t) + \varepsilon g(t, x, u)$ a.e. in $t \geq 0$, $(t, x, u) \in M$.

3. EXISTENCE OF OPTIMAL SOLUTIONS

The existence of a strongly optimal solution is insured by the following result of Baum [3].

THEOREM 1. *Let* A *and* M *be closed, let f be a Carathéodory function and assume that g is a normal integrand satisfying the growth condition* (γ). *If the sets* $\tilde{Q}(t, x)$ *are nonempty convex and satisfy* (K), *if there exists an* $\alpha \in E^1$ *such that* $\Omega_\alpha = \{\{x, u\} \in \Omega : J[x, u] \leq \alpha\} \neq \varnothing$, *and if there exists an* L*-integrable function* ϕ *such that, for all* $\{x, u\} \in \Omega_\alpha$, $g(t, x(t), u(t)) \geq \phi(t)$ *a.e. on* $[0, +\infty)$; *there exists a strongly optimal solution for the problem* $\mathscr{P}$.

To establish existence results for the concepts (ii) to (iv) of optimality in Definition 1, one seeks an appropriate auxiliary problem to which Theorem 1 can be applied. In the case of catching up optimality, such an auxiliary problem is obtained by replacing the original functional with one equivalent to it in the sense of Carathéodory. That is, we consider the functional

$$\mathrm{I}[x, u] = \int_0^{+\infty} h(t, x(t), u(t))\,dt, \tag{7}$$

where $h(t, x, u) = g(t, x, u) - S_t(t, x) - S_x(t, x) \cdot f(t, x, u)$ for a given differentiable function $S : A \to E^1$ satisfying (i) $\limsup_{t \to +\infty} S(t, x(t)) < +\infty$ for all admissible x, (ii) there is an $L \in E^1$ such that $\lim_{t \to +\infty} S(t, x(t)) = L$ for all $\{x, u\} \in \Omega$ with $\mathrm{I}[x, u] < +\infty$, and (iii) $h(t, x, u) \geq 0$ a.e. in M. We will refer to the optimization problem with the functional (7) replacing (2) as $\mathscr{P}^*$. The relationship between problems $\mathscr{P}$ and $\mathscr{P}^*$ is given by the next result.

THEOREM 2. *Let* A *and* M *be closed, f be a Carathéodory function, and g be a normal integrand. Then if* $\{x^*, u^*\}$ *is a strongly optimal solution of* $\mathscr{P}^*$, *it is a catching up optimal solution of* $\mathscr{P}$.

This result is established by showing that the asymptotic properties of $S(t, x(t))$ given above imply, in the case that $\mathrm{I}(x, u) < +\infty$, that $\liminf_{T \to +\infty} \Delta(T) \geq 0$, while $\liminf_{T \to +\infty} \Delta(T) = +\infty$ if $\mathrm{I}(x, u) = +\infty$.

We now combine the first two theorems to obtain the following existence result for catching up optimal solutions.

THEOREM 3. *Let* A *and* M *be closed, let f be a Carathéodory function and assume that g is a normal integrand satisfying the growth condition* (γ).

Further let S *satisfy the conditions* (i) *to* (iii) *given above such that the map* $(t, x) \to S_t(t, x) + S_x(t, x)$ *is a Carathéodory function. Then if there exists* $\alpha \in E^1$ *such that* $\Omega^*_\alpha = \{\{x, u\} \in \Omega : \mathrm{I}[x, u] \leq \alpha\} \neq \varnothing$ *and if the sets* $\tilde{Q}(t, x)$ *are nonempty, convex, and satisfy property* (K) *there exists a catching up optimal solution for* $\mathscr{P}$.

This result contains the basic existence results of [4], [9], and [11] as is shown in [5; 3.3C]. In particular, we show there that the notion of supported trajectories used by those authors can be viewed in terms of Carathéodory's notion of equivalence.

In order to establish the existence of finitely optimal solutions, we consider a sequence $T_n \to +\infty$ and corresponding finite horizon problems on $[0, T_n)$. Here, we require, first a lower closure theorem, which generalizes that of Cesari [6], and which guarantees that a limit of solutions to the finite horizon problems will be a finitely optimal solution to $\mathscr{P}$. In addition, we need the following continuous dependence property for the finite horizon problem.

DEFINITION 3. The control system is said to satisfy the continuous dependence property (C) on $[0, T]$ provided for any sequence $a_n \to a_0$ in E^n, and for any sequence of optimal solutions $\{x^n\}$ of the finite horizon problems

$$(\mathscr{P}_n) \quad \text{minimize} \left\{ \int_0^T g(t, x(t), u(t)\, dt : \{x, u\} \text{ is admissible, } x(T) = a_n \right\}$$

which converges pointwise on $[0, T]$ to an admissible trajectory x^o, with corresponding control u^o, one has $\{x^0, u^o\}$ is an optimal solution for $\mathscr{P}_0$.

While the problem $\mathscr{P}$ exhibits this continuous dependence property for each $T > 0$ under the strong convexity requirements of [4], [9], and [11] (see Remark 1) significantly milder conditions suffice as we have shown in [5; 3.5A]. When property (C) holds we can establish the following existence theorem for finitely optimal solutions.

THEOREM 4. *Let* A *and* M *be closed, f be a Carathéodory function and g be a normal integrand satisfying the growth conditon* (γ). *Assume further that there exists a locally integrable function ψ with $|g(t, x, u)| \leq \psi(t)$ a.e. on $[0, +\infty)$, $(t, x, u) \in M$, and that the sets $\tilde{Q}(t, x)$ are convex and satisfy property* (K). *Then if $\mathscr{P}$ satisfies condition* (C) *for each* $T > 0$, *$\mathscr{P}$ has a finitely optimal solution.*

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