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Analisi matematica. — *Nonlinear analysis. New arguments and results.* Nota I (*) del Socio straniero LAMBERTO CESARI.

RIASSUNTO. — Si presentano condizioni sufficienti in forma astratta per l'esistenza di soluzioni di equazioni operazionali non lineari la cui parte lineare non è autoaggiunta.

1. INTRODUCTION

Recent results of Cesari-Bowman [2] on non selfadjoint non linear problems for ordinary differential equations, of Cesari-Pucci [4b] on non selfadjoint non linear problems for elliptic differential equations, and of Cesari-Kannan [3d] and Cesari-Pucci [4a] for hyperbolic problems were obtained by certain new arguments. We unify here the main points of the arguments in a slightly more general situation, to obtain existence theorems for solutions of operator equations (4.i), (7.i), (7.ii) and corollaries. In particular we formulate, for non selfadjoint problems, some sufficient conditions of the Landesman-Lazer type for existence of solutions.

2. THE ALTERNATIVE METHOD

Let us consider the operational equation

$$Ex = Nx \quad , \quad x \in X, \tag{1}$$

where $E : D(E) \subset X \rightarrow Y$, $N : X \rightarrow Y$ are operators from a Banach space X into a Banach space Y, E linear with domain $D(E) \subset X$, possibly non selfadjoint, with ker E possibly non trivial (resonance), and N not necessarily linear. Usually, E is a linear differential operator in a bounded domain G of $\mathbf{R}^\nu$, $\nu \geq 1$, with associated *linear* homogeneous boundary conditions. Let $P : X \rightarrow X$, $Q : Y \rightarrow Y$ be projection operators (i.e., continuous, linear, idempotent) with $X_0 = PX \supset \ker E$, $X_1 = (I - P) X$, $Y_0 = QY \supset \ker E^*$ where E^* is the adjoint of E , $Y_1 = (I - Q) Y$, where Y_1 is the range of E restricted to $X_1 \cap D(E)$. Thus, we have the decompositions $X = X_0 + X_1$, $Y = Y_0 + Y_1$ (direct sums), and since E is one-one and onto from $X_1 \cap D(E)$ to Y_1, the in-

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verse operator $H : Y_\perp \to X_\perp \cap D(E)$ exists as a linear operator. The following relations usually hold:

(k_1) $H(I-Q)E = I-P$, (k_2) $QE = EP$, (k_3) $EH(I-Q) = I-Q$.

Under these assumptions, then (1) is equivalent to the system of auxiliary and bifurcation equations

$$x = Px + H(I-Q)Nx, \tag{2}$$

$$Q(E-N)x = 0. \tag{3}$$

We refer for details to Cesari [e.g., labcde], and particularly to [1e] where it is mentioned how this system of equations can be related to fixed point theorems and other important tools of analysis. For applications we refer to the same papers, and particulary to [1e] also for references to the many authors who have used this process. As was mentioned in [1e], the bifurcation equation (3) expresses in a global abstract form that process of "casting out the secular terms" which Laplace used step by step in problems of perturbations. The decomposition (2, 3) has been used in problems with strong nonlinearities by many authors (cf. [1e]). Note that, for $X_0 = \ker E$, then relation (k_2) reduces to $QE = 0 = EP$, and equation (3) to $QNx = 0$. While $X_0 = \ker E$ in most applications, the choice of X_0 larger that ker E has been relevant in a number of applications, as in the direct proof in [7] of the fundamental theorems for linear ordinary differential equations in the complex field (Cauchy, Frobenius, Perron, Lettenmeier), in the proof that it is always possible to make H a contraction map ([1a] for Hilbert spaces, [6] for Banach spaces), in the proof [1d] that the use of finite elements for problems at resonance can be framed in the alternative method, and in the recent proof concerning Dirichlet series solutions of differential equations [5].

3. The transformations S and σ

Let $p = \dim X_0$, $q = \dim Y_0$. We shall need the following assumption:

(*) (a) $\infty \geq p \geq q \geq 0$ with a decomposition $X_0 = X_{01} + X_{02}$, $\dim X_{01} = q$; (b) There are continuous maps $\sigma : X_{01} \to Y$, $S : Y_0 \to X_{01}$ such that $S^{-1}(0) = 0$ and $SQ\sigma : X_{01} \to X_{01}$ is the identity map.

First $\dim X_{02} = p - q$ if $\infty > p > q \geq 0$, $\dim X_{02} = \infty$ if $\infty = p > q \geq 0$, and X_{02} istrivial if $\infty > p = q \geq 0$. Now, under assumption (*), problem (1), hence system (2), (3), is equivalent to the fixed point problem for the transformation T, or $(x_{01}, x_{02}, x_1) \to (\bar{x}_{01}, \bar{x}_{02}, \bar{x}_1)$, defined by

$$T \begin{cases} T_1 : \bar{x}_1 = H(I-Q)N(x_{01} + x_{02} + x_1), \\ T_2 : \bar{x}_{01} = x_{01} + k\,SQ(E-N)(x_{01} + x_{02} + x_1), \\ T_3 : \bar{x}_{02} = x_{02}, \end{cases} \tag{4}$$

where k is a positive constant, and $x = x_{01} + x_{02} + x_1$, $\bar{x} = \bar{x}_{01} + \bar{x}_{02} + \bar{x}_1$, $x_{01}, \bar{x}_{01} \in X_{01}$, $x_{02}, \bar{x}_{02} \in X_{02}$, $x_1, \bar{x}_1 \in X_1$. Actually, we shall keep x_{02} fixed in X_{02} (thus, $x_{02} = 0$ if $\infty > p = q \geq 0$), so that T reduces to a map $(x_{01}, x_1) \rightarrow$ $\rightarrow (\bar{x}_{01}, \bar{x}_1)$ defined by the first two relations in (4). Moreover, we shall rewrite T_2 in the following form

$$(5) \qquad T_2 : \bar{x}_{01} = (x_{01} - SQ\,\sigma\, x_{01}) + SQ\,(\sigma x_{01} + k\,(E - N)\, x_{01}) - k\, SQ\,[(E - N)\, x_{01} - (E - N)\,(x_{01} + x_{02} + x_1)\,,$$

where the first term in the second member is zero.

Concerning assumption (*) we note that, whenever $X = Y$, or at least $X_{01} \subset Y$, we can always take σ to be the inclusion map $j : X_{01} \rightarrow Y$. If we assume $\infty > p \geq q \geq 0$, $X = Y = L_2$ (with inner product (u, v) and norm $u = (u, u)^{1/2}$), we can take orthonormal bases in X_0 and Y_0, say $X_0 = sp\,(\phi_1, \ldots, \phi_p)$, $Y_0 = sp\,(\omega_1 \ldots, \omega_q)$, and assume that we may take the bases in such a way that the $q \times q$ matrix $M = [(\omega_s, \phi_i)\,,\ s, i = 1, \ldots, q]$ is non singular. Then we take $X_{01} = sp\,(\phi_1, \ldots, \phi_q)$, $X_{02} = sp\,(\phi_{q+1}, \ldots, \phi_p)$, ($X_{02}$ trivial if $p = q$), and we may define $S : Y_0 \rightarrow X_0$ as follows: For $y \in Y_0$, or $y = \Sigma_s d_s^* \,\omega_s$ with $d^* = \mathrm{col}\,(d_1^*, \ldots, d_q^*)$, $d_s^* = (y, \omega_s)$, take $x = Sy = \Sigma_i d_i\, \phi_i$ with $d = \mathrm{col}\,(d_1, \ldots, d_q)$, $d = M^{-1} d^*$. Then obviously, $S^{-1}(0) = 0$ since M is non singular. On the other hand, if $x \in X_{01} \subset Y$, $x = \Sigma_i c_i \phi_i$, $c = \mathrm{col}$ $(c_1, \ldots, c_q)$ then

$$Qx = \Sigma_s (x, \omega_s)\, \omega_s(t) = \Sigma_s (\Sigma_i c_i \phi_i, \omega_s)\, \omega_s(t) = \Sigma_s (\Sigma_i (\omega_s, \phi_i)\, c_i)\, \omega_s(t) = \Sigma_s c_s^*\, \omega_s(t)\,,$$

where $c^* = \mathrm{col}\,(c_1^*, \ldots, c_q^*)$, $c^* = Mc$. Thus, $SQ\,x = \Sigma_i c_i' \phi_i$, $c' = \mathrm{col}$ $(c_1', \ldots, c_q')$, $c' = M^{-1}(Mc) = c$, and $SQ\sigma$ is the identity map on X_{01}. Note that above we have $Sy = M^{-1} y$.

4. AN EXISTENCE THEOREM

For the sake of simplicity we assume here that $X_0 = \ker E$, $Y_0 = \ker E^*$, so that equation (3) reduces to $QNx = 0$, and T_2 reduces to $\bar{x}_{01} = x_{01} - k\, SQNx$. Let L, d, h be the norms of H, SQ, I $-$ Q, or at least constants such that $\| Hy \|_X \leq L \| y \|_Y$ for $y \in Y_1$, and $\| SQ\, y \|_X \leq d \| y \|_Y$, $\| (I - Q)\, y \|_Y \leq$ $\leq h \| y \|_Y$ for $y \in Y$, and assume that assumption (*) holds.

(4.i) THEOREM. *Let* $C, D, R_0, r, \rho > 0$ and $r' \geq 0$ *be constants such that*

$$\| Nx \|_Y \leq C \quad \textit{for all} \quad x \in X,\ \| x \|_X \leq R_0 + r + r'\,,$$

$$\| Nx - Ny \|_Y \leq D \| x - y \|_X \quad \textit{for all} \quad x, y \in X,\ \| x \|_X,\ \| y \|_X \leq R_0 + r + r'\,,$$

$$\| \sigma x_{01} - k\, N x_{01} \|_Y \leq \rho\, R_0 \qquad \textit{for all} \quad x_{01} \in X_{01}\,,\ \| x_{01} \|_X \leq R_0\,,$$
$$LhC \leq r\,,\ \rho d < 1\,,\ kD(r + r') \leq (1 - \rho d)\, R_0\,.$$

Assume that T *is a compact map from* X *into* X. *Then problem* $Ex = Nx$ *has at least one solution* $x \in X$, $\| x \| \leq R_0 + r + r'$. *Actually,* $r' = 0$ *if* $p = q$; *and if* $p > q$, $r' > 0$, *then the problem has at least one solution* x *for every* $x_{02} \in X_{02}$, $\| x_{02} \| \leq r'$.

Proof. Let S_{01}, S_1 denote the balls in X_{01}, X_1 of center the origin and radii R_0, r respectively. Let x_{02} be an element of X_{02} with $\| x_{02} \| \leq r'$ ($x_{02} = 0$ if $r' = 0$). Let $\Omega = S_{01} \times \{x_{02}\} \times S_1$. Let T denote the transformation defined by (4) on $S_{01} \times \{x_{02}\} \times S_1$ (with $E - N$ replaced by $-N$. Then, for every pair $(x_{01}, x_1) \in S_{01} \times S_1$ we have

$$\| \bar{x}_1 \|_X = \| H(I - Q)\, N(x_{01} + x_{02} + x_1) \|_X \leq LhC \leq r\,,$$
$$\| \bar{x}_{01} \|_X = \| x_{01} - k\, SQN(x_{01} + x_{02} + x_1) \|_X$$
$$= \| (x_{01} - SQ\, \sigma x_{01}) + SQ(\sigma x_{01} - kNx_{01}) + kSQ(Nx_{01} - N(x_{01} + x_{02} + x_1) \|_X$$
$$\leq 0 + \rho d R_0 + k d D(r + r') \leq R_0\,.$$

Thus, T maps $S_{01} \times \{x_{02}\} \times S_1$ into itself. Since T is compact, by Schauder's fixed point theorem, T has at least one fixed point $(x_{01}, x_{02}, x_1) = T(x_{01}, x_{02}, x_1)$ in $S_{01} \times \{x_{02}\} \times S_1$, that is, $x = x_{01} + x_{02} + x_1$ is a solution of (2), (3), hence of (1).

If Y_0 is finite dimensional, so is X_{01}, and if T_1 is known to be compact, then T_2 has finite dimensional range, hence T_2 also is compact, and so is T.

As a particular case we assume now that $X = Y$ is a space of bounded vector functions on a bounded domain G of $\mathbf{R}^\nu$, $\nu \geq 1$, with values in $\mathbf{R}^s$, $s \geq 1$, and $\| x \|_X = \| x \|_\infty = \text{Sup}\,[\,| x(t) |\,,\ t \in G]$, where $|\ \ |$ is a norm in $\mathbf{R}^s$. We assume that N is of the form $Nx = f(t) + g(t, x(t))$, $t \in G$, $x \in X$, with $f : G \to \mathbf{R}^s$, $g : G \times \mathbf{R}^s \to \mathbf{R}^s$, both f and g bounded.

Let $\infty > p \geq q \geq 0$, assume that M is not singular, and let $S : X_{01} \to Y_{01}$ be defined as in no. 3. Let L, d, h be constants such that $\| Hy \|_\infty \leq L \| y \|_\infty$ for all $y \in Y_1$, and $\| SQy \|_\infty \leq d \| y \|_\infty$, $\| (I - Q)\, y \|_\infty \leq h \| y \|_\infty$ for all $y \in Y$. We take for σ the identity map.

(4.ii) COROLLARY. Let $\infty > p \geq q > 0$ *and let* c, C, D, R_0, $r > 0$ *and* $r' \geq 0$ *be constants such that*

$$\| f \|_\infty \leq c\,,\ | g(t, x) | \leq C \qquad \textit{for all} \quad t \in G\,, x \in \mathbf{R}^s,\ | x | \leq R_0 + r + r',$$
$$| g(t, x) - g(t, y) | \leq D\, | x - y | \qquad \textit{for all} \quad t \in G,\ \ x, y \in \mathbf{R}^s,$$
$$| x |\,,\ | y | \leq R_0 + r + r'\,,$$
$$| x - kg(t, x) | \leq \rho R_0 \qquad \textit{for all} \quad x \in \mathbf{R}^s,\ | x | \leq R_0\,, t \in G\,,$$
$$Lh(c + C) \leq r\,,\ \rho d < 1\,,\ kdc + kdD(r + r') \leq (1 - \rho d)\, R_0\,.$$

If T_1 *is compact, then there it at least one solution* x *of* $Ex = Nx$ *with* $|x| \leq R_0 + r + r'$. *Actually,* $r' = 0$ *if* $p = q$; *and if* $p > q$, $r' > 0$, *then the problem has at least one solution* x *for every* $x_{02} \in X_{02}$, $\| x_{02} \|_\infty \leq r'$.

Proof. The proof is the same as for the theorem, where relations (8) are now replaced by

$$\| \bar{x}_1 \|_\infty \leq \| H(I - Q)[f(t) + g(t, x(t))] \|_\infty \leq Lh(c + C) \leq r,$$

$$\| \bar{x}_{01} \|_\infty \leq \| (x_{01} - SQ\sigma x_{01}) + SQ(\sigma x_{01} - kg(t, x_{01}) + kSQ[g(t, x_{01}) -$$

$$- g(t, x_{01} + x_{02} + x_1)] - k\, SQ f(t) \| \leq 0 + d\rho R_0 + kdD(r + r') + kdc \leq R_0 .$$

5. DIFFERENT TOPOLOGIES

In [2], [4b], [4a], [3d] we considered various situations where points of the arguments above could be used by suitable choices of the spaces X, Y.

(*a*) In [2] we considered non selfadjoint problems for ordinary differential equations, say $Ex = x^{(n)} + \Sigma_j p_j(t)\, x^{(n-j)} = f(t) + g(t, x(t))$, $t \in [0, a] \subset \mathbf{R}$, p_j of class C^{n-j}, with linear homogeneous boundary conditions involving $x^{(j)}(0)$, $x^{(j)}(a)$, $j = 0, 1, \ldots, n-1$. Here $f : [0, a] \to \mathbf{R}$, $g : [0, a] \times \mathbf{R} \to \mathbf{R}$ are continuous bounded functions. By first taking $X = Y = L_2[0, a]$, we defined P and Q as the natural orthogonal projection operators onto $X_0 = \ker E$, $Y_0 = \ker E^*$, of dimensions p, q, $\infty > p \geq q \geq 0$, we took σ the identity operator, and we defined S as in no. 3. Then we restricted X, Y to $X^* = Y^* = C[0, a]$, and then $T_1 : X_1 \to X_1$ is compact in the topology of C since its range is contained in $C^n[0, a]$ $n \geq 1$. The corollary applies.

(*b*) In [4b] we considered non selfadjoint elliptic equations of order $2m$, $m \geq 1$, $Ex = Nx$ in a number of situations. In any case, with $X = Y = L_2(G)$, $G \subset \mathbf{R}^\nu$, $\nu \geq 1$, P and Q could be defined as the natural orthogonal projections of X and Y onto $X_0 = \ker E$ and $Y_0 = \ker E^*$ respectively, of dimensions $\infty \geq p \geq q \geq 0$, q assumed to be finite. We assumed further that S could be defined as in no. 3 with M non singular. Let $Nx = f(t) + g(t, Dx)$, $t \in G$, with bounded functions $f : G \to \mathbf{R}$, $g : G \times \mathbf{R}^\mu \to \mathbf{R}$, g continuous, where g depends on t and on the system Dx of the μ derivatives of x in G of orders $0 \leq |\alpha| \leq k_0 < m$.

Assume that the linear homogeneous boundary conditions are expressed in terms of partial derivatives of orders $0 \leq |\alpha| \leq k_0$. Then we considered a space Z, $W \subset H^m \subset Z \subset H^{k_0} \subset L_2(G)$, such that the imbedding maps $j_1 : W \to Z$, $j_2 : Z \to H^{k_0}$ are continuous, and j_1 is compact, and we took $X^* = Y^* = Z$. Then T_1 as a map from X_1 to X_1 is compact since the range of T_1 is in W, and the theorem applies for weak solutions. We called Z the intermediate space.

For instance, for $2(m-k_0) > \nu$, by Sobolev imbedding theorem all elements $x \in H^m$ have distributional partial derivatives $D^\alpha x$, $0 \leq |\alpha| \leq k_0$, all bound d functions in G (and continuous in the interior of G). In this situation, for $Z = H^{k_0}$, or $Z = W^{k_0,\infty}(G)$, the theorem applies with $X^* = Y^* = Z$. As a further particular case, for $k_0 = 0$, and $2m > \nu$, all elements $x \in H^m(G)$ are bounded functions in G. Then for $X^* = Y^* = Z = L_\infty(G)$ the corollary applies.

(*c*) In [3d] and [4a] we considered certain self-adjoint hyperbolic problems in $\mathbf{R}^2$ with periodicity conditions reducing the problem to an interval G in $\mathbf{R}^2$. We took for $X = Y$ a suitable space of continuous functions on G, and for P and Q different projection operators of X and Y onto $X_0 = \ker E$, $Y_0 = \ker E^*$, $E = E^*$, of infinite dimension. Again we had $Nx = f(t, s) + g(t, s, x(t, s))$ with f and g bounded, and T_2 was not compact in C. We then restricted X_0 to a subset X_0^* of X_0, X_0^* convex and closed in C, made up of Lipschitzian functions on G, and such that T_2 maps X_0^* into X_0^*. Now both T_1 and T_2 are compact on X_1 and X_0^* respectively, and T is compact on $X_1 \times X_0$.

Remark 1. For numerical examples of problems mentioned in parts (a), (b), (c) above we refer to the same paper [2], [4b], [4a], [3d].

Remark 2. In [4b], and hence in part (b) above, it is not necessary that E be elliptic. All that is needed is that decompositions $X = X_0 + X_1$, $Y = Y_0 + Y_1$ occur with projection operators P, Q so that $PX = X_0 = \ker E$, $QY = Y_0 = \ker E^*$, $\infty \geq p \geq q \geq 0$, q finite, with $p = \dim X_0$, $q = \dim Y_0$, and a subspace X_{01} of X_0 of dimension q, maps σ and S as in no. 3, and H so as (k_{123}) hold. Examples of this situation will be exhibited elsewhere.

Remark 3. For self adjoint elliptic problems of order $2m$, say

$$(Ex)(t) = f(t) + g(x(t)), \quad t \in G, \quad x \in H,$$

with E elliptic and self adjoint, and $\ker E = \ker E^* = sp(\phi_1, \ldots, \phi_q)$, $g: \mathbf{R} \to \mathbf{R}$, g continuous, with finite limits $g(+\infty)$ and $g(-\infty)$, and for every element $w \in \ker E$, let G^+, G^- denote the subsets of G where $w \geq 0$ and $w \leq 0$ respectively. Then Landesman and Lazer [8] proved that the relation

$$\int_G fw\,dt + g(+\infty)\int_{G^+} |w|\,dt - g(-\infty)\int_{G^-} |w|\,dt > 0 \ (\text{or} < 0)$$
$$\text{for every } w \in \ker E, \ w \not\equiv 0,$$

is a sufficient condition for $Ex = f + g(x)$ to have a solution $x \in H_0^m$. Their proof for $m = 1$ was extended by Williams [11b] to any m, and was motivated by the alternative method. While we referred to a great many extensions in [1e], we only mention here that Shaw [10] extended the statement above to non-selfadjoint elliptic problems under the restriction that ker $E = sp(\phi_1, \dots, \phi_q)$ and ker $E^* = sp(\omega_1, \dots, \omega_q)$ have the same dimension and that the bases can be chosen in such a way that every element $w = \Sigma_i c_i \phi_i$ and corresponding element $\omega = \Sigma c_i \omega_i$ share the same regions of positivity and negativity in G, that is, $w(t)\,\omega(t) \geq 0$ in G.

We also mention in connection with the Landesman-Lazer theorem that, if the values of $g(x)$ lie in the interval $[g(-\infty), g(+\infty)]$, then the condition above with $\geq$ replacing $>$ is a necessary condition for the problem $Ex = f + g(x)$ to have a solution.