Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Marino Badiale, Francis J. Sullivan

A Note on Badiale's Characterization of the q-Gamma Functions

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **76** (1984), n.6, p. 327–329.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1984_8_76_6_327_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1984.

RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta del 15 giugno 1984

Presiede il Presidente della Classe GIUSEPPE MONTALENTI

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *A Note on Badiale's Characterization of the q-Gamma Functions.* Nota di MARINO BADIALE e FRANCIS J. SULLIVAN, presentata (*) dal Socio G. SCORZA DRAGONI.

RIASSUNTO. — Si precisano alcuni risultati del lavoro accennato nel titolo.

In [1] and [2] the first author proved a number of results characterizing the q-gamma functions by functional equations. Recall that

$$\Gamma_p(x) = \begin{cases} (1-q)^{1-x} (q\,;q)_\infty / (q^x\,;q)_\infty & \text{if } 0 \leq q < 1 \\ q^{\binom{x}{2}} (q^{-1}\,;q^{-1})_\infty (q-1)^{1-x} / (q^{-x}\,;q^{-1})_\infty & \text{if } q > 1\,. \end{cases}$$

In particular $\Gamma_0(x) \equiv 1$, and since $\Gamma_q(x) \to \Gamma(x)$ as $q \to 1$ we set $\Gamma_1(x) = \Gamma(x)$.

In analogy with results of Chapter 6 of [0] Badiale established the following: (Corollary to Proposition 2, Page 50 of [2]).

Let $f(q, x)$ be a positive continuous real valued function defined for $0 \leq q \leq 1$, $0 < x$ such that $\mathrm{d}f/\mathrm{d}x$ is continuous and such that

$$f(q, x+1) = ((1-q^x)/(1-q))\, f(q, x) \tag{1.1}$$

(*) Nella seduta del 15 giugno 1984.

and

$$(2.1) \qquad f(q, nx) f(q^n, 1/n) \ldots f(q^n, (n-1)/n)$$
$$= f(q^n, x) f(q^n, x + 1/n) \ldots f\left(q^n, x + \frac{n-1)}{n}\right. (1 + q + \ldots + q^{n-1})^{nx-1}$$

for some positive integer n and all $q < 1$. $n \geq 2$.

Then $f(q, x) = k_q \Gamma_q(x)$ for some k_q depending only on q.

The main purpose of this brief note is to point out that in fact k_q is an absolute constant independent of q. This gives an even stronger analogy with the classical results of Artin for $\Gamma(x)$.

The proof is similar to the last part of the proof of Proposition 1 of (1). Indeed, replacing $f(q, x)$ with $k_q \Gamma_q(x)$ and $f(q^n, x)$ with $k_{q^n} \Gamma_{q^n}(x)$ in (2.1) gives, after simplification:

$k_q k_{q^n}^{n-1} = k_{q^n}^n$. Now $k_{q^n} \neq 0$ by positivity of $f(q, x)$, so cancellation gives $k_q = k_{q^n}$. Iteration then shows $k_q = k_{q^n} = \ldots = k_{q^{n^j}} = \ldots$. Hence by continuity of he quotient $f(q, x)/\Gamma_q(x) = k_q$, as $j \to \infty$, $q^{n^j} \to 0$ and $k_q = k_{q^{n^j}} \to k_0$. Thus $k_q = k_0$ for any $q < 1$, and since $k_q = k_{q^{1/n}} = \ldots$ w- also have $k_q = k_1$, again by continuity. This proves that k_q is independent of q.

A very minor improvement is also possible in Proposition 1 of [1]. This result is notable since it imposes no requirement of continuity in q on the function $f(q, x)$, but rather positivity of $f(q, x)$ together with other conditions.

In this case if we restrict our attention to the domain $0 < q < 1$ (or to the domain $q > 1$) we may weaken the condition (1.3) of [1]

$$f(q, x) > 0 \text{ for all } q \text{ and all } x$$

to

$$(1.3)^* \qquad f(q, x) \neq 0 \text{ for all } q \text{ and all } x.$$

In fact, for $0 < q < 1$ condition (1.3)* together with the existence of d^2f/dx^2, (1.2) and (1.4) of [1] imply (1.3). One sees this as follows. We use the terminology of [1]. Condition (1.4), that is, the fact that $\lim_{q \to 1} f(q, x_0) = \Gamma(x_0)$ for some x_0 and the known positivity of $\Gamma(x)$ give a $q_0 < 1$ such that $f(q, x_0) > 0$ for $1 > q \geq q_0$. Existence of d^2f/dx^2 shows that $f(q, x)$ is of constant sign for q fixed and variable x, so that $f(q, x) > 0$ for $1 > q \geq q_0$ and all x. Then (1.2) of [1] (which is our (2.1) with $n = 2$) gives $f(q^2, \frac{1}{2}) > 0$ whenever $f(q, x) > 0$, so that $f(q^2, x) > 0$ whenever $f(q, x) > 0$. Hence $f(q, x) > 0$ for all q, $1 > q > 0$ and all $x > 0$, and similarly one may treat the domain $q > 1$. For $q = 1$, it follows from Artin's characterization of $\Gamma(x)$ that $f(1, x) = \pm \Gamma_1(x)$.

REFERENCES

[0] ARTIN E. (1964) – *The Gamma Function*, Holt, Rinehart and Winston, New York.
[1] BADIALE M. (1984) – *Characterization of the q–Gamma Functions by Functional Equation* , I, « This journal », vol. LXXIV, pp. 7-11.
[2] BADIALE M. (1983) – *Characterization of the q-gamma Functions by Functional Equations*, II, « This journal », vol. LXXIV, pp. 49-54.