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Meccanica dei solidi. — *On the dynamical behaviour of plates in unilateral contact with an elastic foundation: a finite element approach.*
Nota di LUIGI ASCIONE, DOMENICO BRUNO e RENATO S. OLIVITO, presentata (*) dal Corrisp. E. GIANGRECO.

RIASSUNTO. — In questo lavoro viene studiato il comportamento dinamico di una piastra vincolata monolateralmente su una fondazione elastica alla Winkler. Si presentano alcuni risultati numerici ottenuti mediante discretizzazione agli elementi finiti. Tali risultati mettono in luce l'influenza di alcuni fattori tipici come le funzioni di forma, il parametro di mesh e l'ampiezza dell'intervallo con cui si realizza l'integrazione nel tempo delle equazioni del moto.

Si istituiscono infine dei confronti con risultati numerici ottenuti precedentemente da altri autori.

1. INTRODUCTION

The unilateral contact problems are a subject of great interest both in theoretical and applied mechanics.

In recent years much research has been done in this context.

Mainly, the static contact problems have been investigated and significant results, from a theoretical and a numerical point of view, can be found in literature [1-2-3-4-5].

In this framework the contact problems involving beams or plates resting on tensionless elastic foundations represent a subject of relevant structural interest. More specifically, the static unbonded contact on a Winkler subgrade has been analyzed in [6-7-8-9-10], while results relative to the elastic half-space model can be found in [11-12-13].

On the contrary, only a few numerical investigations have been developed in the dynamical field [14-15-16].

Some mathematical aspects of the problem are examined in the basic works [2-3].

The aim of the present paper is a numerical analysis relative to the dynamical problem of an elastic plate in unilateral contact with a Winkler subgrade.

The plate is modelled according to Mindlin's theory. In this way the effects of the shear stresses and of the rotatory inertia on the motion can

(*) Nella seduta dell'11 febbraio 1984.

be taken into account. Further on, this model facilitates the use of C^0-elements [7].

A variational formulation of the problem is presented and a theorem of existence and uniqueness of the dynamical solution is given.

Some numerical results, obtained by using a finite-element discretization are discussed. The time integration of the non linear motion equations is achieved via Newmark's method.

An evaluation of the influence of some factors such as the interpolation functions, the mesh parameter and the time stepsize is possible through these results.

Finally, a comparison with some previous results by D. Talaslidis and P.D. Panagiotopoulos [18] is presented.

2. Formulation of the problem

In this section a variational formulation of the dynamical problem of a rectangular plate resting on a tensionless and frictionless elastic foundation (fig. 1) is given.

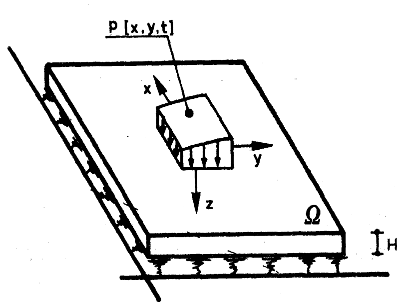


Fig. 1.

This formulation falls within the class of problems examined in [2-3], for which results of existence and uniqueness of the solution have been proven.

2.1. *The plate and elastic foundation models.*

According to Mindlin's theory [19], the displacement field components of the plate can be expressed by the following:

$$u_1 = - z\psi_x (x , y , t) , \tag{2.1a}$$

$$u_2 = - z\psi_y (x , y , t) , \tag{2.1b}$$

$$u_3 = w (x , y , t) , \tag{2.1c}$$

where ψ_x and ψ_y are the bending slopes along the x and y axes.

Eqs. (2.1) differ from the corresponding ones in the classical thin plate theory in that functions ψ_i yet to be determined replace $w,_i$ in u_1 and u_2. Therefore, it is assumed that line elements originally normal to the midplane remain straight on the deformation (no warping), but the assumption that such line elements remain normal to the midplane, after deformation, is abandoned.

The strain components for the assumed displacement field follow immediately as:

$$\tag{2.2} \begin{aligned} &\varepsilon_{xx} = - z\psi_{x,x}, &&\varepsilon_{yy} = - z\psi_{y,y}, &&\varepsilon_{zz} = 0, \\ &\varepsilon_{xy} = - \frac{z}{2}(\psi_{x,y} + \psi_{y,x}), &&\varepsilon_{xz} = \frac{1}{2}(w,_x - \psi_x), &&\varepsilon_{yz} = \frac{1}{2}(w,_y - \psi_y). \end{aligned}$$

Assuming that the normal stress σ_{zz} can be neglected with respect to the other ones, it is possible to express the stress components from eqs. (2.2) by means of Hooke's law:

$$\tag{2.3} \begin{aligned} &\sigma_{xx} = \frac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\varepsilon_{yy}), &&\sigma_{yy} = \frac{E}{1-\nu^2}(\varepsilon_{yy} + \nu\varepsilon_{xx}), &&\sigma_{zz} \simeq 0, \\ &\sigma_{xy} = 2\,G\,\varepsilon_{xy}, &&\sigma_{xz} = 2\,G\,\varepsilon_{xz}, &&\sigma_{yz} = 2\,G\,\varepsilon_{yz}. \end{aligned}$$

In eqs. (2.3) E and G are the plate elastic moduli, while ν is the Poisson's ratio.

As far as the elastic foundation is concerned, it is assumed that it consists of a continuous distribution of massless elastic springs, which cannot react in tension. Where the contact is active, the spring reaction r is proportional to the plate deflection w, according to Winkler's assumption, i.e.:

$$\tag{2.4} r = - K_f\, w^+$$

where K_f is a positive constant (Winkler modulus) and w^+ is the positive part of w ($w^+ = \max\{w, 0\}$).

2.2. *Variational formulation.*

The plate dynamical equilibrium equations can be expressed by means of the virtual work principle:

$$\tag{2.5} \begin{aligned} \iint_{\Omega}\int_{-H/2}^{+H/2} \sigma_{ij}\,\delta\varepsilon_{ij}\,d\Omega\,dz = \iint_{\Omega} p\,\delta w\,d\Omega - \iint_{\Omega} K_f\,w^+\,\delta w\,d\Omega\, - \\ - \iint_{\Omega}\int_{-H/2}^{+H/2} \rho\ddot{u}_i\,\delta\ddot{u}_i\,d\Omega\,dz \end{aligned}$$

where ρ is the material density, the dot represents the time derivation and δu_i are the virtual displacement field components.

From eqs. (2.2) and (2.3), eq. (2.5) can be rewritten in the following form:

$$(2.6)\qquad D\iint_\Omega [\psi_{x,x}\,\delta\psi_{x,x}+\psi_{y,y}\,\delta\psi_{y,y}+\nu\,(\psi_{x,x}\,\delta\psi_{y,y}+\psi_{y,y}\,\delta\psi_{x,x})]\,d\Omega\,+$$

$$+\frac{GH^3}{12}\iint_\Omega [(\psi_{x,y}+\psi_{y,x})\,(\delta\psi_{x,y}+\delta\psi_{y,x})]\,d\Omega+\chi GH\iint_\Omega [(w_{,x}-\psi_x)\,(\delta w_{,x}-\delta\psi_x)\,+$$

$$+(w_{,y}-\psi_y)\,(\delta w_{,y}-\delta\psi_y)]\,d\Omega=\iint_\Omega p\,\delta w\,d\Omega-\iint_\Omega K_f\,w^+\,\delta w\,d\Omega\,+$$

$$-\iint\left(\frac{\rho H^3}{12}\ddot{\psi}_x\,\delta\psi_x+\frac{\rho H^3}{12}\ddot{\psi}_y\,\delta\psi_y+\rho H\,\ddot{w}\,\delta w\right)d\Omega$$

where D is the plate flexural stiffness and χ is the shear correction factor.

The closed subspace of $(H^1(\Omega))^3$ which consists of the plate admissible displacements (w, ψ_x, ψ_y) is denoted by V_0 ($H^1(\Omega)$ is the usual Sobolev space of order one on the open region Ω). Evidently, if no rigid support exists, V_0 coincides with the whole space $(H^1(\Omega))^3$.

Furthermore, let the following be:

$\mathscr{H}=(L^2(\Omega))^3$	($L^2(\Omega)$ is the space of square summable functions on Ω),
V_0'	the dual space of V_0,
X	a generic Banach space with norm $\|\cdot\|_X$,
$L^p(0, T, X)$	the Banach space of the mesurable functions $t\in[0,T]\to\to f(t)\in X$, such that:

$$(2.7)\qquad \left(\int_0^T \|f(t)\|_X^p\,dt\right)^{1/p}=\|f\|_{L^p(0,T;X)}<\infty\qquad (p<\infty).$$

$$(2.8)\qquad \|f\|_{L^\infty(0,T;X)}=\sup_{t\in[0,T]}\text{ess}\ \|f(t)\|_X\,.$$

The following theorem holds:

THEOREM. *It is assumed that*:

$$(2.9)\qquad p\,,\,\dot{p}\in L^2(0\,,T\,;V_0')\,,$$

$$(2.10)\qquad (w_0\,,\psi_{x0}\,,\psi_{y0})\in V_0\,,\,(w_1\,,\psi_{x1}\,,\psi_{y1})\in\mathscr{H}.$$

There exists a unique element $(w, \psi_x, \psi_y) \in V_0$ such that:

(2.11a) $$(w, \psi_x, \psi_y) \in L^\infty(0, T; V_0),$$

(2.11a) $$(\dot{w}, \dot{\psi}_x, \dot{\psi}_y) \in L^\infty(0, T; \mathscr{H}),$$

(2.11a) $$(\ddot{w}, \ddot{\psi}_x, \ddot{\psi}_y) \in L^\infty(0, T; V_0'),$$

and satisfying eq. (2.6) *with the initial conditions:*

(2.12a) $$w(x, y, 0) = w_0(x, y), \qquad \psi_x(x, y, 0) = \psi_{x0}(x, y),$$
$$\psi_y(x, y, 0) = \psi_{y0}(x, y),$$

(2.12b) $$\dot{w}(x, y, 0) = w_1(x, y), \qquad \dot{\psi}_x(x, y, 0) = \psi_{x1}(x, y),$$
$$\dot{\psi}_y(x, y, 0) = \psi_{y1}(x, y).$$

For proof of the above theorem reference to [2] is made.

3. NUMERICAL TREATMENT OF THE VARIATIONAL PROBLEM

The region Ω is discretized by means of a family of rectangular finite elements $\{\Omega_e\}_{e=1}^{N_e}$ $(\bar{\Omega} = \bigcup_{e=1}^{N_e} \bar{\Omega}_e, \ \Omega_e \cap \Omega_f = \varnothing$ if $e \neq f)$ and let $f_i(x, y)$ be the global interpolation functions. In this way, for each $t \geq 0$, a subspace V_0^h of V_0 (h is the usual mesh parameter) is identified: it is spanned by the system of the N_G linearly independent functions $\{(f_i, f_i, f_i)\}_{i=1}^{N_G}$, that is supposed complete in V_0 (four and eight node isoparametric rectangular elements will be considered).

An approximate solution of the problem (2.6) can be taken as:

(3.1a) $$w^h(x, y, t) = \sum_{i=1}^{N_G} w_i(t) f_i(x, y),$$

(3.1b) $$\psi_x^h(x, y, t) = \sum_{i=1}^{N_G} \psi_{xi}(t) f_i(x, y),$$

(3.1c) $$\psi_y^h(x, y, t) = \sum_{i=1}^{N_G} \psi_{yi}(t) f_i(x, y),$$

where $w_i, \psi_{xi}, \psi_{yi}$ are the nodal values of the functions w^h, ψ_x^h, ψ_y^h.

Consequently, taking into account eqs. (3.1), the motion equation (2.6) can be put in the following discrete form:

(3.2) $$M\ddot{u} + K(u)\, u = p,$$

where M and K (u) are respectively the mass and stiffness matrices of the dynamical system, p and u are respectively the vectors of the generalized loads and displacements. Obviously, the stiffness matrix K (u) is a nonlinear function of u because of the second term on the right hand of the eq. (2.6).

By simple algebra the following is obtained:

$$p=[p_i]=\left[\iint_\Omega p f_i \,\mathrm{d}\Omega\right], \tag{3.3}$$

$$\mathrm{M}=\begin{bmatrix} \mathrm{M}^{(w)} & & \\ & \mathrm{M}^{(\Psi)} & \\ & & \mathrm{M}^{(\Psi)} \end{bmatrix}, \qquad \mathrm{K}=\begin{bmatrix} \mathrm{K}^{(w,w)} & \mathrm{K}^{(w,x)} & \mathrm{K}^{(w,y)} \\ & \mathrm{K}^{(x,x)} & \mathrm{K}^{(x,y)} \\ \mathrm{sym} & & \mathrm{K}^{(y,y)} \end{bmatrix} \tag{3.4}$$

where:

$$\mathrm{M}_{ij}^{(w)}=\iint_\Omega \rho \mathrm{H} f_i f_j \,\mathrm{d}\Omega \tag{3.5a}$$

$$\mathrm{M}_{ij}^{(\Psi)}\iint_\Omega \frac{\rho \mathrm{H}^3}{12} f_i f_j \,\mathrm{d}\Omega \tag{3.5b}$$

$$\mathrm{K}_{ij}^{(y,y)}=\mathrm{D}\left[\iint_\Omega f_{i,y} f_{j,x} \,\mathrm{d}\Omega+\frac{1-\nu}{2}\iint_\Omega f_{i,x} f_{j,y} \,\mathrm{d}\Omega\right]+\chi \mathrm{GH}\iint_\Omega f_i f_j \,\mathrm{d}\Omega \tag{3.6a}$$

$$\mathrm{K}_{ij}^{(x,x)}=\mathrm{D}\left[\iint f_{i,x} f_{j,y} \,\mathrm{d}\Omega+\frac{1-\nu}{2}\iint_\Omega f_{i,y} f_{j,x} \,\mathrm{d}\Omega\right]+\chi \mathrm{GH}\iint_\Omega f_i f_j \,\mathrm{d}\Omega \tag{3.6b}$$

$$\mathrm{K}_{ij}^{(x,y)}=\mathrm{D}\left[\nu\iint_\Omega f_{i,x} f_{j,y} \,\mathrm{d}\Omega+\frac{1-\nu}{2}\iint_\Omega f_{i,y} f_{j,x} \,\mathrm{d}\Omega\right] \tag{3.6c}$$

$$\mathrm{K}_{ij}^{(w,x)}=-\chi \mathrm{GH}\iint_\Omega f_{i,x} f_j \,\mathrm{d}\Omega \tag{3.6d}$$

$$\mathrm{K}_{ij}^{(w,y)}=-\chi \mathrm{GH}\iint_\Omega f_{i,y} f_j \,\mathrm{d}\Omega \tag{3.6e}$$

$$\mathrm{K}_{ij}^{(w,w)}=\bar{\mathrm{K}}_{ij}^{(w,w)}+\tilde{\mathrm{K}}_{ij}^{(w,w)} \tag{3.7a}$$

$$\bar{\mathrm{K}}_{ij}^{(w,w)}=\chi \mathrm{GH}\left[\iint_\Omega f_{j,x} f_{j,x} \,\mathrm{d}\Omega+\iint_\Omega f_{i,y} f_{j,y} \,\mathrm{d}\Omega\right] \tag{3.7b}$$

$$\tilde{\mathrm{K}}_{ij}^{(w,w)}=\sum_{e=1}^{\mathrm{N}_e}\sum_{\gamma_e=1}^{\mathrm{G}_e} \mathrm{K}_f \mathrm{P}_{\gamma e} f_i(x_{\gamma e}, y_{\gamma e}) f_j(x_{\gamma e}, y_{\gamma e}) \tag{3.7c}$$

In the last equation the sum is extended to all N_e finite elements and G_e gaussian points of coordinates $(x_{\gamma e}, y_{\gamma e})$ on the e-th element. The coefficients $P_{\gamma e}$ are defined as:

$$
\text{(3.8)} \qquad P_{\gamma e} = \begin{cases} W_{\gamma e} \text{ (Gaussian weight at the point } (x_{\gamma e}, y_{\gamma e}) \\ \qquad \text{if } \sum\limits_{i=1}^{N_G} w_i f_i (x_{\gamma e}, y_{\gamma e}) \leq 0) \\ 0 \quad \left(\text{if } \sum\limits_{i=1}^{N_G} w_i f_i (x_{\gamma e}, y_{\gamma e}) < 0\right) \end{cases}
$$

Eq. (3.2) can be solved, for each time t, by means of the following iterative procedure:

$$
\text{(3.9)} \qquad M\ddot{u}^{(k)} + K(u^{(k-1)})\, u^{(k)} = p \,,
$$

i.e. at the k—th step eq. (3.9) is solved by evaluating the matrix $K^{(w,w)}$ from eqs. (3.8) for $u = u^{(k-1)}$: the first step corresponds to the bilateral contact problem.

4. NUMERICAL APPLICATIONS AND CONCLUDING REMARKS

In this last section some numerical results regarding the dynamical equilibrium problem of a rectangular plate subject to the loads shown in fig. 2 and to zero initial conditions are presented.

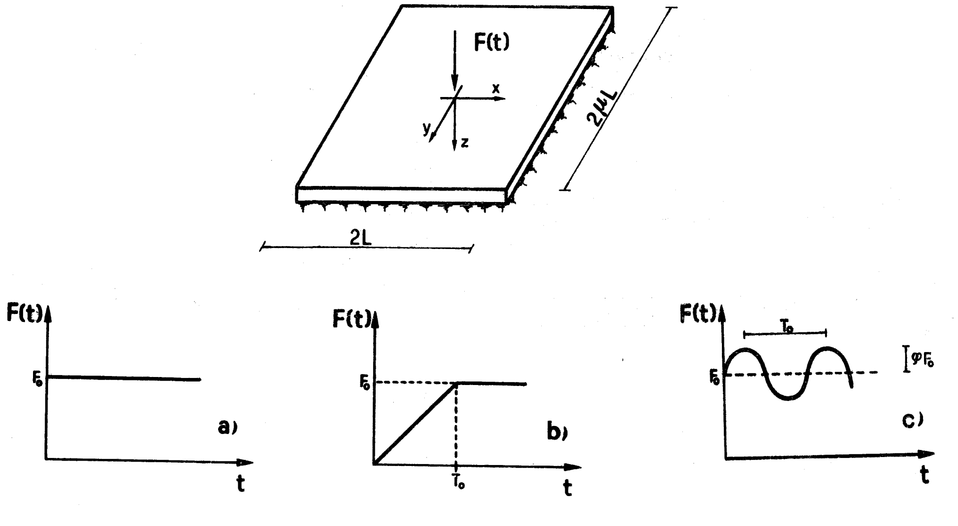


Fig. 2. – Load conditions.

The time integration is carried out by means of the classical Newmark's method with $\delta = 0.5$ and $\alpha = 0.25$ [20].

The presence of the unilateral constraints does not permit the formulation of a classical eigenvalue problem and consequently the time integration can be only carried out by means of direct methods.

In order to deal with dimensionless quantities the following is assumed:

$$\xi = x/L\,, \qquad \eta = y/\mu L\,, \tag{4.1a}$$

$$\tau = t/t_0\,, \qquad t_0 = L^2\left(\frac{\rho H}{D}\right)^{1/2}, \tag{4.1b}$$

$$\varphi_\xi = \frac{D}{F_0 L}\,\psi_x\,, \qquad \varphi_\eta = \frac{D}{F_0 L}\,\psi_y\,, \qquad v = \frac{D}{F_0 L^2}\,w\,. \tag{4.1c}$$

By substituting eqs. (4.1) into eq. (2.6), it is easy to show that, for an assigned load condition, the dynamical solution depends only on the four parameters which follow:

$$\mu\,, \qquad \Delta = \frac{H}{L}\,, \qquad \Gamma = \frac{K_f L}{F}\,, \qquad \nu\,. \tag{4.2}$$

EXAMPLE 1. The classical problem of a beam resting on an elastic foundation is analyzed. The load condition consists in a suddenly applied load acting at the middle section (fig. 2a).

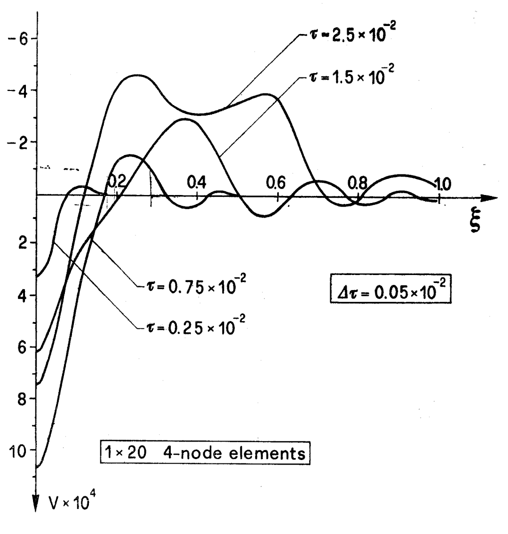


Fig. 3 *a*.

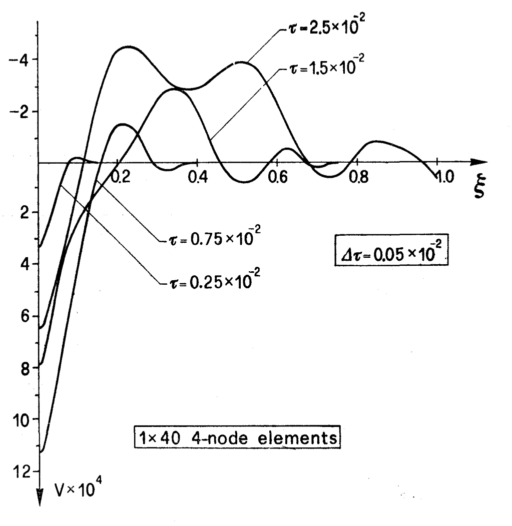


Fig. 3 *b*

The parameters used are:

$$\mu = 0.05 \,, \qquad \Delta = 0.05 \,, \qquad \Gamma = 1 \,, \qquad \nu = 0.1 \tag{4.3}$$

Figs. 3*a*-*b*-*c* show some numerical results for the dimensionless transverse deflection v along the beam axis; in particular they correspond to uniform meshes of $(1 \times)$ 20, 40, 60 four node rectangular elements (due to the symmetry of the scheme, only one-half of the beam has been discretized) and to some fixed values of the dimensionless time parameter τ. The time stepsize used in the Newmark's scheme is $\Delta\tau = 0.05 \times 10^{-2}$.

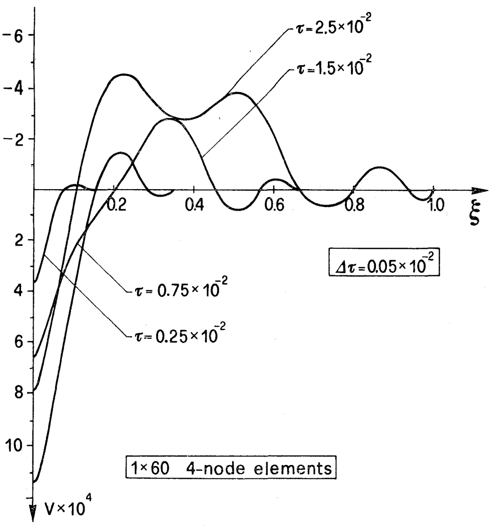


Fig. 3 *c*.

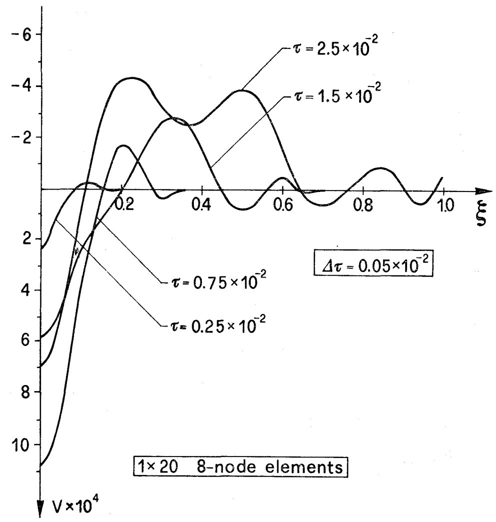


Fig. 3 *d*.

On the contrary, the numerical results corresponding to a uniform mesh of $(1 \times)$ 20 eight-node rectangular elements are shown in fig. 3 *d*.

The influence of the time stepsize is analyzed in figs. 4 *a*-*b*, where the sections $\xi = 0$ and $\xi = 0.15$ are examined. The sections correspond to the two different situations:

i) continuous contact with the subgrade,

ii) transition from active to inactive states of the unilateral springs.

For each time step, the iterative scheme proposed in Sec. 3 to solve the non linear dynamical equilibrium equations converges in a few steps (only $4 \simeq 6$ steps).

EXAMPLE 2. This example refers to the previously examined beam, which is now subject to the load conditions shown in figs. 2 *b-c* ($\varphi = 0.1$, $T_0/t_0 = 2.5$), consisting in a linearly increasing force and in a pulsating one respectively, acting at the middle section.

The discretization is achieved by means of a uniform mesh of ($1 \times$) 40 four-node rectangular elements (on one-half of the beam).

The time stepsize used in Newmark's scheme is the same as in the previous example.

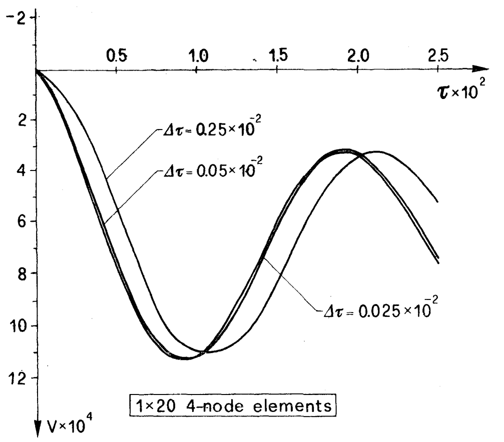


Fig. 4 *a*.

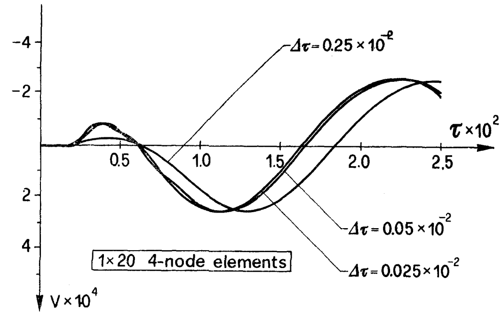


Fig. 4 *b*.

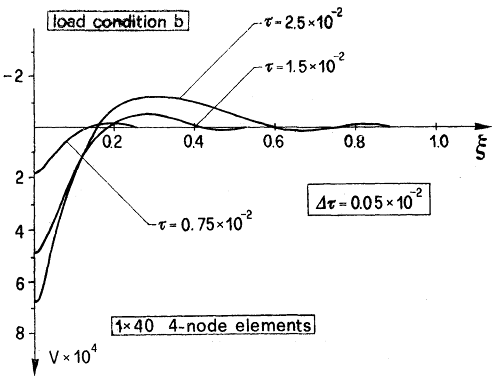


Fig. 5 *a*.

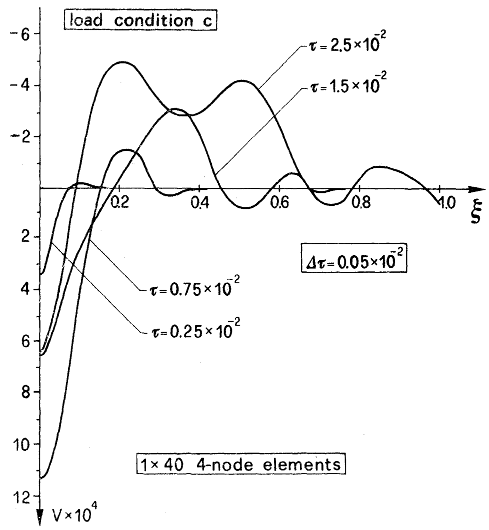


Fig. 5 *b*.

EXAMPLE 3. A two-dimensional problem corresponding to a square plate subjected a suddenly applied force at the centre is now examined. The parameters used are as follows:

$$\mu = 1\,, \qquad \Delta = 0.05\,, \qquad \Gamma = 1\,, \qquad \nu = 0.1\,. \tag{4.4}$$

Due to the biaxial symmetry of the scheme only one-quarter of the plate as been discretized.

Figs. 6 *a-b* show some numerical results obtained by using different meshes of four and eight node rectangular elements: the dimensionless transverse deflections v along the ξ axis is plotted against the dimensionless abscissa ξ for some fixed values of the parameter τ. The time stepsize used in the Newmark's scheme is $\Delta\tau = 0.05 \times 10^{-2}$.

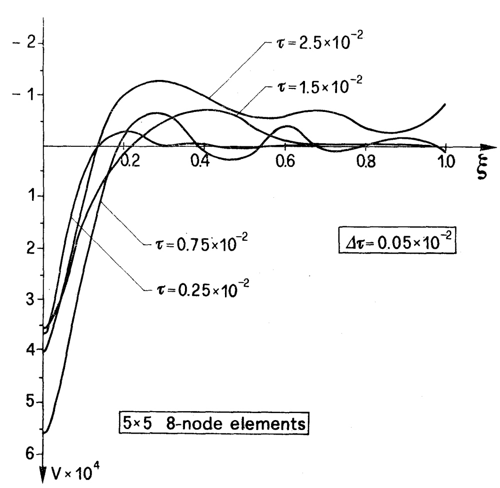


Fig. 6 *a*.

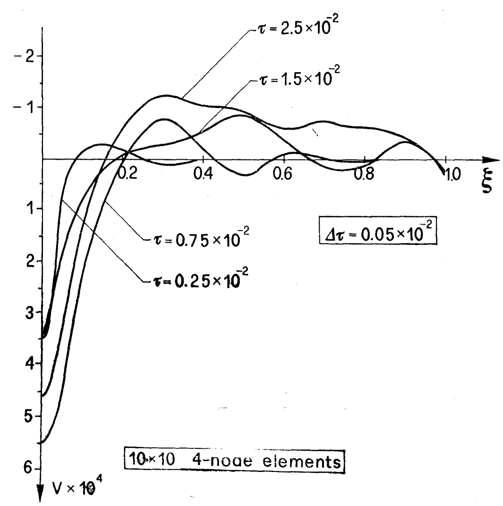


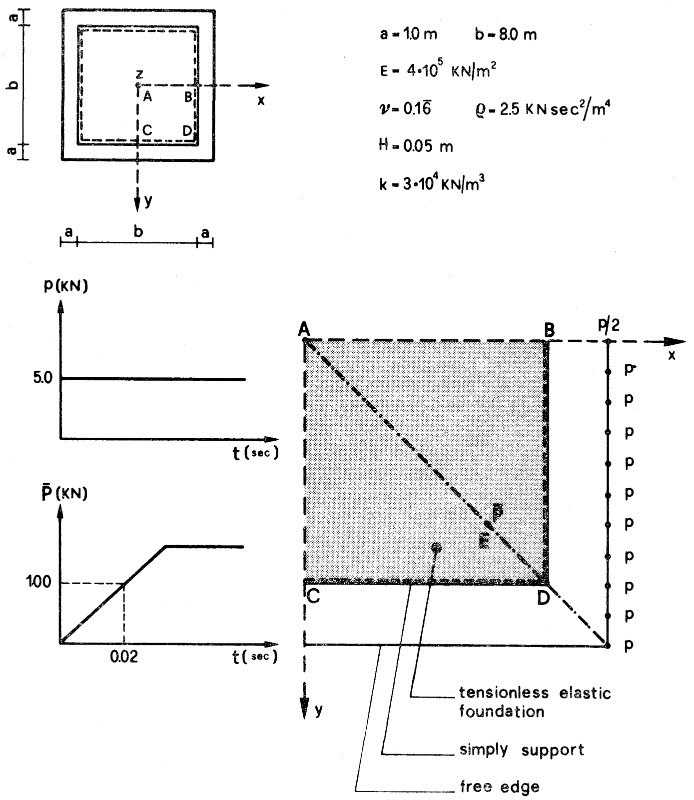
Fig. 6 *b*.

EXAMPLE 4. This last example has been analyzed in a recent paper by Talaslidis and Panagiotopoulos [18] (fig. 7).

The suddenly applied forces act upon the edges and the centre of the plate. Furthermore, a linearly increasing force $\overline{P}$ is applied at the node E. The plate analyzed in [18] corresponds to Kirchhoff's model and the finite element discretization for one-quarter of it utilizes 308 and 64 degrees-of-freedom and constraints, respectively.

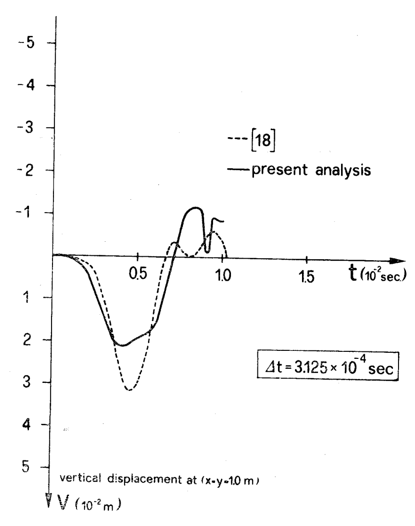
Since in this case the rigid body motions are avoided by the simple support, a reduced integration of the shear terms in eq. (2.6) has been carried out [21, 22]: a full integration gives, indeed, completely erroneous results.

On the contrary, the numerical results obtained in the previous examples (for which the rigid body motions can occur) correspond to a full integration of these terms.



a = 1.0 m b = 8.0 m
E = 4·10^5 KN/m^2
ν = 0.16
ρ = 2.5 KN sec^2/m^4
H = 0.05 m
k = 3·10^4 KN/m^3
P (KN)
5.0
t (sec)
P̄ (KN)
100
0.02
A
B
C
D
E
x
y
z
P/2
P
tensionless elastic foundation
simply support
free edge

Fig. 7.



---[18]
—present analysis
0.5
1.0
1.5
t (10^-2 sec)
Δt = 3.125 × 10^-4 sec
vertical displacement at (x=y=1.0 m)
V (10^-2 m)

Fig. 8.

Finally, a comparison between the numerical results given in [18] and the present ones, corresponding to a finite element mesh of 15×15 four-node rectangular elements, is presented in fig. 8.

A good qualitative agreement between results is noted, although the numerical values are quite different. In this respect, it is observed that under the same conditions Mindlin's model is more flexible than Kirchhoff's; consequently, the relative stiffness plate-foundation decreases and this agrees with the smallest value of the maximum displacement exhibited in our analysis at the node E.

Aknowledgement

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