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Magnetofluidodinamica. — *The Lagrangian and Hamiltonian formulations for the waves in a compressible fluid with the Hall current.*

Nota di GIULIO MATTEI (*), presentata (**) dal Socio D. GRAFFI.

RIASSUNTO. — In questo lavoro si ricavano: 1) l'equazione d'onda linearizzata, 2) la formulazione Lagrangiana, 3) la formulazione Hamiltoniana, nella teoria della propagazione ondosa in un fluido comprimibile descritto dalle equazioni della magnetofluidodinamica ideale in presenza di corrente Hall.

1. INTRODUCTION

In this paper we give: (i) the linearized wave equation, (ii) the Lagrangian formulation, (iii) the Hamiltonian formulation, in the theory of wave propagation in a fluid described by the equations of ideal magnetofluiddynamics (MFD) in the presence of the Hall current. In a preceding paper [1] on the same subject the fluid has been considered incompressible; the present paper expands the matter of [1] to the case of the compressible fluid. As in [1], we adopt the terminology « Hall current fluid » (HCF) introduced in [2].

2. THE LINEARIZED WAVE EQUATION FOR A COMPRESSIBLE HCF

For a compressible HCF the basic linearized equations are (Gaussian units)

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\frac{a_0^2}{\rho_0}\,\mathrm{grad}\,\rho + \frac{1}{4\pi\mu\rho_0}\,(\mathrm{curl}\,\boldsymbol{b}) \wedge \mathbf{B}_0 \tag{2.1}$$

$$\frac{\partial \rho}{\partial t} = -\rho_0\,\mathrm{div}\,\boldsymbol{v} \tag{2.2}$$

$$\frac{\partial \boldsymbol{b}}{\partial t} = \mathrm{curl}\,(\boldsymbol{v} \wedge \mathbf{B}_0) + \beta\,\mathrm{curl}\,(\mathbf{B}_0 \wedge \mathrm{curl}\,\boldsymbol{b}) \tag{2.3}$$

$$\mathrm{div}\,\boldsymbol{b} = 0\,, \tag{2.3'}$$

where the quantities with the subscript o refer to the unperturbed (constant) state and $\boldsymbol{v}$, ρ , $\boldsymbol{b}/\mu$ are the perturbations in velocity, density and magnetic field,

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μ is the (constant) magnetic permeability, a_0 is the (constant) speed of sound and β is given by

$$\beta = \frac{c^2 \beta_H}{4 \pi \mu} ; \tag{2.4}$$

in (2.4) c is the speed of light in a vacuum and β_H is the Hall coefficient

Taking the curl of (2.1) and using (2.3), we obtain

$$\frac{\partial \boldsymbol{q}}{\partial t} = \text{curl} (\boldsymbol{v} \wedge \mathbf{B}_0) \tag{2.5}$$

where

$$\boldsymbol{q} = \boldsymbol{b} + 4 \pi \mu \rho_0 \beta \omega \qquad (\omega = \text{curl}\, \boldsymbol{v}) . \tag{2.6}$$

In the linear approximation we may write the velocity $\boldsymbol{v}$ in the form

$$\boldsymbol{v} = \frac{\partial \boldsymbol{s}}{\partial t} , \tag{2.7}$$

where $\boldsymbol{s} = \boldsymbol{s} (x_1 , x_2 , x_3 , t)$ is the displacement of a fluid element from its equilibrium position. Substituting in (2.5) and integrating, we obtain

$$\boldsymbol{q} = \text{curl} (\boldsymbol{s} \wedge \mathbf{B}_0) \tag{2.8}$$

so that

$$\boldsymbol{b} = \text{curl} (\boldsymbol{s} \wedge \mathbf{B}_0) - 4 \pi \mu \rho_0 \beta \omega \qquad \left(\omega = \frac{\partial}{\partial t} \text{curl}\, \boldsymbol{s}\right) . \tag{2.9}$$

From (2.2) we obtain

$$\rho = - \rho_0 \, \text{div}\, \boldsymbol{s} . \tag{2.10}$$

Using (2.7), (2.9), (2.10), introducing the *Alfvén velocity* $\mathbf{A}_0 = \mathbf{B}_0 / 4 \pi \mu \rho_0$ and taking the unit vector $\boldsymbol{e}_3$ of the x_3-axis parallel to $\mathbf{B}_0$, we obtain from (2.1) the three-dimensional wave equation for the compressible HCF

$$\left(\frac{\partial^2}{\partial t^2} - a_0^2 \, \text{grad div}\right) \boldsymbol{s} - \mathrm{A}_0^2 \, [\text{curl curl} (\boldsymbol{s} \wedge \boldsymbol{e}_3)] \wedge \boldsymbol{e}_3 - \tag{2.11}$$
$$- \beta \mathrm{B}_0 \, \boldsymbol{e}_3 \wedge \text{curl}\, \omega = 0 .$$

It will be useful to write (2.9) in the form

$$b = -\mathbf{B}_0 \operatorname{div} s + \mathrm{B}_0 \frac{\partial s}{\partial x_3} - 4\pi \mu \rho_0 \beta \omega \tag{2.9'}$$

easily obtainable using known formulas of vector analysis.

If we are interested in the propagation in an arbitrary direction of harmonic plane waves characterized by a frequency $p/2\pi$ and a wave-number vector $\boldsymbol{k}$, we may choose the frame of reference in such a way that

$$\mathbf{B}_0 \equiv (0, 0, \mathrm{B}_0) \quad , \quad \boldsymbol{k} \equiv (0, k_2, k_3)$$

and then, using standard developments, we obtain from (2.11) the dispersion equation

$$[p^2 - \mathrm{A}_0^2 k^2 \cos^2\theta] \{ p^4 - k^2 (a_0^2 + \mathrm{A}_0^2) p^2 + a_0^2 \mathrm{A}_0^2 k^4 \cos^2\theta \} + \tag{2.12}$$

$$+ \beta^2 k^4 p^2 \mathrm{B}_0^2 \cos^2\theta \, (a_0^2 k^2 - p_0^2) = 0 ,$$

where θ is the angle between $\boldsymbol{k}$ and $\mathbf{B}_0$. Eq. (2.12) is studied in [3] n. 6.1, where it is obtained following a way different from that used here.

3. The Lagrangian formulation for continuous systems

It is well-known (see, for example, [4] Sect. 11-2, [5] Sect. 1, [6] Sect. 45, [7] p. 118) that a spatially homogeneous continuous system in which energy is conserved can be described in terms of a *Lagrangian density* L, which is a function of (say) n generalized coordinates (field variables) $\psi_h(x_k, t)$, together with their first derivatives

$$\dot{\psi}_h \overset{\text{def}}{=} \frac{\partial \psi_h}{\partial t} \qquad \text{and} \qquad \psi_{h,k} \overset{\text{def}}{=} \frac{\partial \psi_h}{\partial x_k} \; (h = 1, 2, \ldots, n; \; k = 1, 2, 3), \tag{3.1}$$

so that

$$\mathrm{L} = \mathrm{L}(\psi_h, \dot{\psi}_h, \psi_{h,k}). \tag{3.2}$$

For each ψ_h there will be an equation of motion (field equation) in the form

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathrm{L}}{\partial \dot{\psi}_h}\right) + \sum_{k=1}^{3} \frac{\partial}{\partial x_k}\left(\frac{\partial \mathrm{L}}{\partial \psi_{h,k}}\right) - \frac{\partial \mathrm{L}}{\partial \psi_h} = 0 \tag{3.3}$$

which can be derived from Hamilton's principle.

4. The Hamiltonian formulation for continuous systems

It is possible (see, for example, [4] Sect. 11-4, [5] Sect. 1, [6] Sect. 45) to give a Hamiltonian formulation for the continuous systems considered in Sect. 3, defining a *canonical momentum density*

$$\pi_h \overset{\text{def}}{=} \frac{\partial L}{\partial \dot{\psi}_h} \tag{4.1}$$

and a *Hamiltonian density*

$$H \overset{\text{def}}{=} \sum_{h=1}^{n} \pi_h \dot{\psi}_h - L . \tag{4.2}$$

The corresponding canonical field equations are

$$\dot{\pi}_h = -\frac{\partial H}{\partial \psi_h} + \sum_{k=1}^{3} \frac{\partial}{\partial x_k}\left(\frac{\partial H}{\partial \psi_{h,k}}\right) \quad ; \quad \dot{\psi}_h = \frac{\partial H}{\partial \pi_h} . \tag{4.3}$$

5. The wave equation for a compressible HCF deduced from a Lagrangian formulation

Our goal is to find the correct Lagrangian density for a compressible HCF leading to the wave equation (2.11).

At first we identify the field variables ψ_h introduced in Sect. 3 with the components s_h $(h = 1, 2, 3)$ of the vector $\boldsymbol{s}$ introduced in Sect. 2. Then consider (see [4] (11.40) and also [8] p. 314) the expression of the Lagrangian density for the sound vibrations in a non-conducting fluid

$$L = \frac{1}{2} \rho_0 [\dot{s}^2 - a_0^2 (\operatorname{div} \boldsymbol{s})^2]$$

where $\dot{\boldsymbol{s}} \overset{\text{def}}{=} \frac{\partial \boldsymbol{s}}{\partial t}$. Using the following facts: (i) the total energy density w for a HCF is given by (see, for example, [2]) $w = w_* + B^2/8 \pi \mu$, where w_* is the energy density for a non-conducting fluid and $\mathbf{B} = \mathbf{B}_0 + \boldsymbol{b}$; (ii) the term

$$\frac{\mathbf{B}_0 \cdot \boldsymbol{b}}{4 \pi \mu} = \rho_0 A_0^2 \left(\frac{\partial s_3}{\partial x_3} - \operatorname{div} \boldsymbol{s}\right) - \rho_0 B_0 \beta \omega_3 \qquad \text{(see (2.9'))}$$

cannot contribute (as well as-obviously- the term $B_0^2/8 \pi\mu$, which is the constant magnetic energy of the unperturbed state) to the equations of motion (see (3.3), being $\dfrac{\partial\,(\operatorname{div} \boldsymbol{s})}{\partial\, s_{h,k}} = \delta_{hk}$); we may write the required Lagrangian density in the form

$$L = \frac{\rho_0}{2}\,[\,\dot{s}^2 - a_0^2\,(\operatorname{div} \boldsymbol{s})^2] - \frac{b^2}{8\,\pi\mu}\,.$$

Using (2.9′) and noting that the term $\omega^2 = (\operatorname{curl} \dot{\boldsymbol{s}})^2$ cannot contribute to the equation of motion (see (3.3)), we may write the Lagrangian density in the final form

$$(5.1)\quad L = \frac{\rho_0}{2}\left[\,\dot{s}^2 - (a_0^2 + A_0^2)\,(\operatorname{div} \boldsymbol{s})^2 - A_0^2\left(\frac{\partial\, \boldsymbol{s}}{\partial\, x_3}\right)^2 + 2\,A_0^2\,\frac{\partial s_3}{\partial x_3}\operatorname{div} \boldsymbol{s}\right] +$$
$$+ B_0\,\beta\,\rho_0\left(\frac{\partial \boldsymbol{s}}{\partial x_3}\,.\,\boldsymbol{\omega} - \omega_3 \operatorname{div} \boldsymbol{s}\right)\,.$$

The Lagrangian density (5.1) does lead exactly to the wave equation (2.11), as may be readily verified using (3.3).

So we have achieved our goal of describing wave propagation in a compressible HCF by a Lagrangian formulation.

6. THE WAVE EQUATION FOR A COMPRESSIBLE HCF DEDUCED FROM A HAMILTONIAN FORMULATION

For a compressible HCF the canonical momentum density (cf. (4.1) and (5.1)) is

$$(6.1)\qquad \pi_h = \rho_0\,\dot{s}_h$$

and the Hamiltonian density is (cf. (4.2), (5.1) and (6.1))

$$(6.2)\qquad H = \frac{1}{2\,\rho_0}\sum_{h=1}^{3} \pi_h^2 - \Phi$$

where

$$(6.3)\quad \Phi = \frac{\rho_0}{2}\left[- (a_0^2 + A_0^2)\,(\operatorname{div} \boldsymbol{s})^2 - A_0^2\left(\frac{\partial \boldsymbol{s}}{\partial x_3}\right)^2 + 2\,A_0^2\,\frac{\partial s_3}{\partial x_3}\operatorname{div} \boldsymbol{s}\right] +$$
$$+ \rho_0\,B_0\,\beta\left(\frac{\partial \boldsymbol{s}}{\partial x_3}\cdot\boldsymbol{\omega} - \omega_3 \operatorname{div} \boldsymbol{s}\right)\,.$$

The corresponding canonical field equations are (cf. (4.3))

$$\dot{s}_h = \pi_h / \rho_0 \ , \tag{6.4}$$

which merely repeats (6.1), and

$$\dot{\pi}_h = - \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \left(\frac{\partial \Phi}{\partial s_{h,k}} \right) , \tag{6.5}$$

with Φ given by (6.3).

It may be readily verified that (6.5), using (6.4), does lead to the wave equation (2.11).

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