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Pontrjagin forms of quaternion manifolds.

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Geometria differenziale. — *Pontrjagin forms of quaternion manifolds*. Nota di VASILE OPROIU, presentata (*) dal Socio E. MARTINELLI.

RIASSUNTO. — Si dimostra che per le varietà a struttura quaternionale generalizzata integrabile, le classi di Pontrjagin sono generate dalle classi di Pontrjagin del fibrato vettoriale fondamentale.

1. A quaternion manifold is a $4m$-dimensional real manifold M carrying an integrable almost quaternal structure. Such a structure was introduced firstly in [4]. It has been shown, [3], that for the quaternion manifolds the quaternion local charts are related by projective co-ordinate transformations. On the other hand, studying the integrability problem in the general frame of the G-structure theory, the author has shown, [6], that the quaternion manifolds have the property that the curvature tensor field of a torsion free adapted connection has a special expression. This expression was obtained studying the family of the torsion free adapted connections and finding an invariant tensor field which resembles the projective or H-projective curvature tensor fields in the real or complex cases. In this paper we show that the Pontrjagin ring of a quaternion manifold is generated by the first Pontrjagin class, which, up to a constant factor, is the same as the first (and only) Pontrjagin class of the fundamental vector bundle. In the case where the dimension is 4 the total Pontrjagin class is trivial. If $m > 1$ there are situations where some combinations of Pontrjagin classes are trivial but, generally, the Pontrjagin classes of a quaternion manifold may be not trivial. The typical examples of quaternion manifolds are the quaternion projective spaces. Remark also the similar results obtained in the case of the quaternion Kaehler manifolds of constant Q-sectional curvature [5].

2. THE CURVATURE INVARIANT.

Let us recall the results of [6]. Consider a $4m$-dimensional manifold M on which it is defined an almost quaternal structure. That means that there exists a (fundamental) vector sub-bundle V, with fibre dimension 3, in the bundle of (1,1)-type tensors on M, having canonical bases, i.e. for every point $p \in \mathrm{M}$ there exists a neighbourhood U of p and 3 tensor fields $\mathrm{F}_1, \mathrm{F}_2, \mathrm{F}_3$ of type (1,1), sections of the restriction of V to U, satisfying:

$$\mathrm{F}_a^2 = -\mathrm{I} \quad ; \quad \mathrm{F}_a \mathrm{F}_b = -\mathrm{F}_b \mathrm{F}_a = \mathrm{F}_c . \tag{1}$$

(*) Nella seduta del 14 gennaio 1984.

Throughout this paper (a, b, c) will be a cyclic permutation of $(1, 2, 3)$. The tensor fields (F_a), (F'_a) defining canonical bases of V on U, U', respectively, are related on $U \cap U'$ by:

$$F'_a = \sum_{b=1}^{3} s_{ab} F_b$$

where $[s_{ab}] \in SO(3)$. The structural group is $G = GL(m, \mathbf{H})$. $Sp(1)$ where $Sp(1) \cong SO(3)$ and $GL(m, \mathbf{H})$ is the (real representation of the) linear group of square non-degenerate matrices of order m with entries quaternions.

If N_a is the Nijehuis tensor field of F_a, let:

$$N = N_1 + N_2 + N_3$$

and let α_a be the 1-form defined by:

$$\alpha_a(X) = \frac{1}{4m\text{-}2} \quad \text{trace}\,(Y \rightarrow F_a N(X, Y)).$$

Remark that:

$$\sum_{a=1}^{3} \alpha_a (F_a X) = 0 .$$

There exists a torsion free adapted connection to the almost quaternal structure if and only if:

$$N(X, Y) = \sum_{a=1}^{3} \{ \alpha_a(Y) F_a X - \alpha_a(X) F_a Y \}. \tag{2}$$

An adapted connection is a linear connection ∇ on M preserving V:

$$\nabla_X F_a = -\eta_b(X) F_c + \eta_c(X) F_b \tag{3}$$

where η_1, η_2, η_3 are locally defined 1-forms associated with ∇.

If ∇ is torsion free, adapted, the family of the torsion free adapted connections is given by the formula:

$$\tilde{\nabla}_X Y = \nabla_X Y + \omega(X) Y + \omega(Y) X - \sum_{a=1}^{3} \{ \omega(F_a X) F_a Y + \omega(F_a Y) F_a X \} \tag{4}$$

where ω is a 1-form, globally defined on M. The corresponding local 1-forms $\tilde{\eta}_a$ are:

$$\tilde{\eta}_a(X) = \eta_a(X) - 2\,\omega(F_a X) .$$

The curvature invariant of the almost quaternal structure is the Weyl Q- projective tensor field corresponding to the transformation (4):

$$(5)\quad W(X,Y)Z = R(X,Y)Z + \{Q(X,Y) - Q(Y,X)\}Z + Q(X,Z)Y -$$

$$- Q(Y,Z)X - \sum_{a=1}^{3}\{Q(X,F_aY) - Q(Y,F_aX)\}F_aZ -$$

$$- \{\sum_{a=1}^{3} Q(X,F_aZ)F_aY - Q(Y,F_aZ)F_aX\}$$

where:

$$(6)\quad Q(X,Y) = \frac{1}{4(m+1)}S(X,Y) + \frac{m+3}{16m(m+1)(m+2)}\{S(X,Y) +$$

$$+ S(Y,X)\} - \frac{1}{16m(m+2)}\sum_{a=1}^{3}\{S(F_aX,F_aY) + S(F_aY,F_aX)\}$$

$S(Y,Z) = \mathrm{trace}\,(X \to R(X,Y)Z) =$ the Ricci tensor field of ∇.

The almost quaternal structure is integrable if and only if N is given by (2) and $W = 0$.

3. THE CURVATURE MATRICES OF QUATERNION MANIFOLDS

PROPOSITION 1. *Let ∇ be a torsion free adapted connection to the almost quaternal structure. Then there exists an adapted torsion free connection with symmetric Ricci tensor field.*

Proof. If $\nabla, \tilde{\nabla}$ are related by (4) then:

$$\tilde{S}(X,Y) - \tilde{S}(Y,X) = S(X,Y) - S(Y,X) - (4m+4)\{(\nabla_X\omega)(Y) - (\nabla_Y\omega)(X)\}$$

Since ∇ is torsion free:

$$(\nabla_X\omega)(Y) - (\nabla_Y\omega)(X) = d\omega(X,Y).$$

Due to the first Bianchi identity we have:

$$S(X,Y) - S(Y,X) = -\,\mathrm{trace}\,R(X,Y).$$

The 2-form trace $R(X,Y)$ is exact since it represents in the de Rham cohomology of M the 2-dimensional Pontrjagin class of M and this class is trivial. Thus:

$$\mathrm{trace}\,R(X,Y) = d\alpha(X,Y)$$

where α is a 1-form globally defined on M. Then:

$$\tilde{S}(X, Y) - \tilde{S}(Y, X) = d[-\alpha - (4m+4)\omega](X, Y).$$

Taking $\omega = \dfrac{-1}{4m+4}\alpha$ the proof is over.

From now on we suppose that the Ricci tensor field of the torsion free adapted connection ∇ is symmetric. Then Q itself is symmetric:

$$\text{(6')} \qquad Q(X, Y) = \frac{2m+3}{8m(m+2)} S(X, Y) - \frac{1}{8m(m+2)} \sum_{a=1}^{3} S(F_a X, F_a Y).$$

Suppose that M is a quaternion manifold. Then:

$$\text{(7)} \quad R(X, Y)Z = -Q(X, Z)Y + Q(Y, Z)X + \sum_{a=1}^{3} \{Q(X, F_a Y) - $$
$$- Q(Y, F_a X)\} F_a Z + \sum_{a=1}^{3} \{Q(X, F_a Z) F_a Y - Q(Y, F_a Z) F_a X\}.$$

Let $(U, x^1, \ldots, x^{4m})$ be a local chart on M and denote by $\Omega = [\Omega^l_k]$ the curvature matrix associated with R in this local chart. The entries of Ω are the curvature 2-forms:

$$\text{(8)} \quad \Omega^l_k = \frac{1}{2} R_{ijk}{}^l dx^i \wedge dx^j = \delta^l_i Q_{jk}\, dx^i \wedge dx^j + Q_{ih} \left\{ \sum_{a=1}^{3} (F_a)^h_j (F_a)^l_k \right\} dx^i \wedge dx^j +$$
$$Q_{ih} \left\{ \sum_{a=1}^{3} (F_a)^l_j (F_a)^h_k \right\} dx^i \wedge dx^j.$$

Denote by F_a the matrices associated with F_a in the given local chart. The following 2-forms, locally defined, are useful:

$$\text{(9)} \qquad \varphi_a = \frac{1}{4m} \quad \text{trace} \quad F_a \Omega.$$

Using the expression (8) of Ω^l_k it follows easily:

$$\text{(9')} \qquad \varphi_a = (F_a)^h_i Q_{hj}\, dx^i \wedge dx^j.$$

PROPOSITION 2. *The 2-forms φ_a are related to the local 1-forms η_a associated with ∇ by the formulae* :

$$\text{(10)} \qquad d\eta_a + \eta_b \wedge \eta_c = -2\varphi_a.$$

Proof. Using the definition $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ it follows easily from (3):

$$R(X, Y) F_a - F_a R(X, Y) = -(d\eta_b + \eta_c \wedge \eta_a)(X, Y) F_c + \\ + (d\eta_c + \eta_a \wedge \eta_b)(X, Y) F_b .$$

On the other hand, using the expression (7) of R we have:

$$(11) \qquad R(X, Y) F_a - F_a R(X, Y) = 2 \varphi_b (X, Y) F_c - 2 \varphi_c (X, Y) F_b .$$

The relation (10) is now obtained easily.

REMARK. Using, the 2-forms φ_a introduced above we obtain:

$$(8') \qquad \Omega_k^l = -\sum_{a=1}^{3} (F_a)_k^l \varphi_a + \delta_i^l Q_{jk} dx^i \wedge dx^j + Q_{ih} \left\{ \sum_{a=1}^{3} (F_a)_j^l (F_a)_k^h \right\} dx^i \wedge dx^j.$$

REMARK. Due to the symmetry of Q we get trace $\Omega = 0$.

The following relation will be useful later:

$$(12) \quad Q_{li} \{ (F_b)_j^h (F_c)_k^l - (F_c)_j^h (F_b)_k^l \} dx^i \wedge dx^j = -(F_a \Omega)_k^h + \varphi_a \delta_k^h - \varphi_b (F_c)_k^h + \\ + \varphi_c (F_b)_k^h + \{(F_a)_i^h Q_{kj} + \delta_i^h Q_{lj} (F_a)_k^l\} dx^i \wedge dx^j .$$

Recall also that the first Bianchi identity satisfied by R is expressed in terms of the curvature matrices Ω as follows:

$$(13) \qquad \Omega_k^l \wedge dx^k = 0 .$$

4. PONTRJAGIN FORMS.

The ring of the Pontrjagin forms of the manifold M can be generated by the closed exterior forms obtained from det $(I + (\sqrt{-1}/2\pi)\Omega)$. The cohomology class of det $(I + (\sqrt{-1}/2\pi)\Omega)$ in the de Rham cohomology of M is the total Pontrjagin class of M. Recall that the same Pontrjagin ring can be generated by classes defined by the closed forms trace Ω, trace $\Omega^2, \ldots,$ trace Ω^{2m}, globally defined on M. The forms trace Ω^{2k+1} are exact on M, thus the corresponding Pontrjagin classes are trivial [1]. To obtain our result we shall use the expression (8) of the curvature matrices and the properties (11), (12).

LEMMA 1. *For every quaternion manifold we have*:

$$(14) \qquad \Omega_l^h \wedge \Omega_k^l = -3 \sum_{a=1}^{3} \varphi_a \wedge (F_a)_l^h \Omega_k^l + 2 \left(\sum_{a=1}^{3} \varphi_a^2 \right) \delta_k^h + \\ + 2 \sum_{a=1}^{3} \varphi_a \wedge \{\delta_i^h (F_a)_k^l Q_{lj} + (F_a)_i^h Q_{kj}\} dx^i \wedge dx^j .$$

Proof. Use the expression (8′) of Ω and the property (13):

$$\Omega^h_l \wedge \Omega^l_k = -\sum_{a=1}^{3} \varphi_a \wedge \Omega^h_l (F_a)^l_k + 2\sum_{a=1}^{3} \varphi_a \wedge \{ (F_c)^h_i Q_{jl} (F_b)^l_k - $$

$$- (F_b)^h_i Q_{jl} (F_c)^l_k \Big\} dx^i \wedge dx^j .$$

Then, using (11) and (12) we get the desired result.

COROLLARY.

$$\text{trace } \Omega^2 = 4(1-m) \sum_{a=1}^{3} \varphi_a^2 . \tag{15}$$

This is obtained easily from the above formula for Ω^2.

It follows that for $m=1$ the total Pontrjagin class of M is 1. In the case of the quaternion projective line $P^1(H)$ this is obvious since it is diffeomorphic to the 4-dimensional sphere S^4.

LEMMA 2. *For every quaternion manifold we have:*

$$\Omega^3 = -3\left(\sum_{a=1}^{3} \varphi_a F_a\right) \wedge \Omega^2 + 2\sum_{a=1}^{3} \varphi_a^2 \wedge \Omega . \tag{16}$$

Proof. Using the result of Lemma 1 we have:

$$(\Omega^3)^h_k = -3\sum_{a=1}^{3} \varphi_a \wedge \Omega^h_l (F_a)^l_r \wedge \Omega^r_k + 2\left(\sum_{a=1}^{3} \varphi_a^2\right) \wedge \Omega^h_k +$$

$$+ 2\sum_{a=1}^{3} \varphi_a \wedge \{ \Omega^h_i \wedge dx^i (F_a)^l_k Q_{lj} + \Omega^h_l (F_a)^l_i \wedge dx^i Q_{kj} \} \wedge dx^j .$$

Then, due to (13) and (11) we get:

$$(\Omega^3)^h_k = -3\sum_{a=1}^{3} \varphi_a \wedge \{(F_a)^h_l \Omega^l_r + 2\varphi_b (F_c)^h_r - 2\varphi_c (F_b)^h_r\} \wedge \Omega^r_k +$$

$$+2\left(\sum_{a=1}^{3} \varphi_a^2\right) \wedge \Omega^h_k + 2\sum_{a=1}^{3} \varphi_a \wedge \{(F_a)^h_l \Omega^l_i + 2\varphi_b (F_c)^h_i - 2\varphi_c (F_b)^h_i\} Q_{kj} dx^i \wedge dx^j$$

Doing the necessary cancellations we obtain the result.

COROLLARY.

$$\text{trace } \Omega^3 = 0 . \tag{17}$$

We have trace $\Omega^3 = - 3 \sum_{a=1}^{3} \varphi_a \wedge \text{trace}\,(F_a \Omega^2) + 2 \left(\sum_{a=1}^{3} \varphi_a^2 \right) \wedge \text{trace}\ \Omega.$

But trace $\Omega = 0$, then, from Lemma 1.

$$(18) \qquad \text{trace}\ F_a \Omega^2 = 3\,\varphi_a \wedge \text{trace}\ \Omega - 3\,\varphi_b \wedge \text{trace}\ F_c \Omega + 3\,\varphi_c \wedge \text{trace}\ F_b \Omega =$$
$$= - 12\, m\, \varphi_b \wedge \varphi_c + 12\, \varphi_c \wedge \varphi_c = 0$$

since trace $F_a = 0$ and $\varphi_b \wedge \varphi_c = \varphi_c \wedge \varphi_b$.

For convenience denote:

$$(19) \qquad \alpha = \sum_{a=1}^{3} \varphi_a F_a .$$

Then α is a vector valued 1-form and it follows easily:

$$(20) \qquad \alpha^2 = - \left(\sum_{a=1}^{3} \varphi_a^2 \right) I .$$

With this notation the result of Lemma 2 can be written as:

$$(16') \qquad \Omega^3 = - 3\, \alpha \wedge \Omega^2 - 2\, \alpha^2 \wedge \Omega .$$

Now we may state our main result:

THEOREM 3. *For every quaternion manifold we have*:

$$\text{trace}\ \Omega^{2k} = (-1)^{k-1}\, 4\, (2^{2k-1} - m - 1) \left(\sum_{a=1}^{3} \varphi_a^2 \right)^k ; \ \text{trace}\ \Omega^{2k+1} = 0 .$$

Proof. First we obtain by induction the following formula for the powers of Ω:

$$(21) \qquad \Omega^k = (-1)^k\, (2^{\,k-1} - 1)\, \alpha^{k-2} \wedge \Omega^2 + (-1)^k\, (2^{k-1} - 2)\, \alpha^{k-1} \wedge \Omega \ ; k \geq 3 .$$

In fact, for $k = 3$, due to (19) we have the result of Lemma 2. If (20) is valid for k, then using (16'):

$$\Omega^{k+1} = (-1)^k\, (2^{k-1} - 1)\, \alpha^{k-2} \wedge (- 3\, \alpha \wedge \Omega^2 - 2\, \alpha^2 \wedge \Omega) + (- 1)^k$$
$$+ (2^{k-1} - 2)\, \alpha^{k-1} \wedge \Omega^2 = (- 1^{k+1} (2^k - 1)\, \alpha^{k-1} \wedge \Omega^2 + (-1)^{k+1} (2^k - 2)\, \alpha^k \wedge \Omega$$

which is (21) for $k \to k + 1$. Next we have:

$$\text{trace}\ \Omega^{2k+1} = - (2^{2k} - 1)\, (-1)^{k-1} \left(\sum_{a=1}^{3} \varphi_a^2 \right)^{k-1} \wedge \left(\sum_{a=1}^{3} \varphi_a \wedge \text{trace}\ F\ \ \Omega^2 \right) -$$
$$- (2^k - 2)\, (-1)^k \left(\sum_{a=1}^{3} \varphi_a^2 \right)^k \wedge \text{trace}\ \Omega = 0$$

due to the corollary to Lemma 2. Then:

$$\text{trace}\quad \Omega^{2k} = (2^{2k-1} - 1)(-1)^{k-1}\left(\sum_{a=1}^{3}\varphi_a^2\right)^{k-1} \wedge \text{trace}\,\Omega^2 +$$
$$+ (2^{2k-1} - 2)(-1)^{k-1}\left(\sum_{a=1}^{3}\varphi_a^2\right)^{k-1} \wedge \left(\sum_{a=1}^{3}\varphi_a \wedge \text{trace}\,F_a\,\Omega\right) =$$
$$= (-1)^{k-1} 4(1-m)(2^{2k-1}-1)\left(\sum_{a=1}^{3}\varphi_a^2\right)^{k} + (-1)^{k-1} 4m(2^{2k-1} -$$
$$- 2)\left(\sum_{a=1}^{3}\varphi_a^2\right)^{k} = (-1)^{k-1} 4(2^{2k-1} - m - 1)\left(\sum_{a=1}^{3}\varphi_a^2\right)^{k}$$

Thus, it follows that the Pontriagin ring of M is generated by the de Rham cohomology class of the fundamental 4-form $\sum_{a=1}^{3}\varphi_a^2$ which, if $m \neq 1$, may be replaced by the first Pontrjagin form trace Ω^2. If $m = 2^{2k-1} - 1$ we obtain the interesting result trace $\Omega^{2k} = 0$, the other forms being not trivial. The corresponding Pontrjagin classes may be not trivial. In the case of the quaternion projective spaces the form $\sum_{a=1}^{3}\varphi_a^2$ is closed but not exact. In fact, $\left(\sum_{a=1}^{3}\varphi_a^2\right)^m$ is the volume $4m$-form of M, multiplied by a constant. Thus, in this case, the Pontrjagin classes are, generally, not trivial.

The usual Pontrjagin classes are represented by the $4k$-forms π_k defined by:

$$1 - \pi_1 + \pi_2 - \pi_3 + \dots = \det\left(I + (\sqrt{-1}/2\pi)\,\Omega\right).$$

The relations with the generators trace Ω^2, trace Ω^4, . . are given by

$$2k\,\pi_k + \sum_{l=1}^{k}(2\pi)^{-2l}\,\pi_{k-l} \wedge \text{trace}\,\Omega^{2l} = 0\,.$$

If $m = 2^{2h-1} - 1$ the forms π_h may be not trivial. The first few forms π_k are:

$$\pi_1 = \frac{1}{2}(2\pi)^{-2}\quad \text{trace}\quad \Omega^2 = -2(2\pi)^{-2}(1-m)\sum_{a=1}^{3}\varphi_a^2$$
$$\pi_2 = \frac{1}{4}\left(2 + \frac{2^3 - m - 1}{(m-1)^2}\right)\pi_1^2$$
$$\pi_3 = \frac{1}{6}\left(1 + \frac{3}{2}\,\frac{2^3 - m - 1}{(m-1)^2} - \frac{2^5 - m - 1}{2(m-1)^3}\right)\pi_1^3\,.$$

Finally, we should obtain the relation between the fundamental 4-form $\sum_{a=1}^{3}\varphi_a^2$ and the 1-forms η_a corresponding to the connection ∇ .

From (10) we get:

$$4 \sum_{a=1}^{3} \varphi_a^2 = \sum_{a=1}^{3} (d\eta_a + \eta_b \wedge \eta_c)^2 = \sum_{a=1}^{3} (d\eta_a^2 + 2\, d\eta_a \wedge \eta_b \wedge \eta_c).$$

Then the 2-forms $d\eta_a + \eta_b \wedge \eta_c$ can be considered as the curvature 2-forms of a linear connection on the vector bundle V and $\sum_{a=1}^{3} (d\eta_a + \eta_b \wedge \eta_c)^2$ is, up to a constant factor, the first Pontrjagin class of V. In fact the connection ∇ defines a linear connection on V, whose local connections forms with respect to the local basis (F_1, F_2, F_3) are components of the matrix:

$$\begin{pmatrix} 0 & \eta_3 & -\eta_2 \\ -\eta_3 & 0 & \eta_1 \\ \eta_2 & -\eta_1 & 0 \end{pmatrix}.$$

The curvature matrix with respect to the same local basis is:

$$\Theta = \begin{pmatrix} 0 & d\eta_3 + \eta_1 \wedge \eta_2 & -(d\eta_2 + \eta_3 \wedge \eta_1) \\ -(d\eta_3 + \eta_1 \wedge \eta_2) & 0 & d\eta_1 + \eta_2 \wedge \eta_3 \\ (d\eta_2 + \eta_3 \wedge \eta_1) & -(d\eta_1 + \eta_2 \wedge \eta_3) & 0 \end{pmatrix}.$$

The first (and only) Pontrjagin class of V is represented, up to constant factor, by the trace Θ^2:

$$\operatorname{trace} \Theta^2 = -2 \sum_{a=1}^{3} (d\eta_a + \eta_b \wedge \eta_c)^2 .$$

Thus the Pontrjagin classes of M are generated by the Pontrjagin class of V.

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