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On the convergence of Neumann series in Banach space.


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RIASSUNTO. — Si estende un risultato di N. Suzuki sulla convergenza della serie di Neumann per un operatore compatto in uno spazio di Banach.

0. Introduction

Let $(S, B, \mu)$ be a measure space with $\mu$ positive and finite. Consider the Fredholm equation

\[ h(x) - \int_S K(x, y) h(y) \, d(y) = f(x) \]

where the kernel $K(\cdot, \cdot)$ is of Hilbert-Schmidt type. One of the oldest iterative methods for solving this equation is related to the so called Neumann series,

\[ f(x) + \sum_{n=1}^{\infty} \int_S K_n(x, y) f(y) \, d\mu(y) \]

where $K_n(\cdot, \cdot)$ is the $n$-th kernel defined by $K(\cdot, \cdot)$. A sufficient condition for the convergence of the Neumann's series is that the spectral radius of the operator defined on $L^2(S, B, \mu)$ by

\[ g(x) = \int_S K(x, y) g(y) \, d\mu(y) \]

be less than 1.

This may be considered as a special case of the following problem: Let $X$ be a complex Banach space and $T$ in $L(X)$, $L(X)$ is the set of all bounded

linear operators defined on $X$ with values in $X$, and consider the equation

\[ x - Tx = y \]

with $y$ given in $X$ and $T$ is supposed to be a compact element in $L(X)$.

A (iterative) scheme for solving the equation (1) is to consider the formal Neumann series

\[ \sum_{n=0}^{\infty} T^n y. \]

It is obvious that if this is a convergent series then the sum gives the solution of the equation (1).

In (4) the following result is proved.

**Theorem 1.** Let $T$ be in $L(X)$ and compact. Then, in order that the Neumann series (2) may be strongly convergent it is necessary and sufficient that

\[ \lim || T^n y || = 0. \]

The purpose of the present Note is to show that there are other classes of operators in $L(X)$ for which a result like that in Theorem 1 is valid.

In order to do this we recall the following definition of a class of (not necessarily linear) mappings.

**Definition 2.** A continuous mapping $f : X \to X$ is said to be a locally power $\alpha$-set contraction if for each non-compact bounded set $M$ in $X$ there exists an integer $n = n(M)$ such that

\[ \alpha (f^n (M)) \leq k \alpha (M) \]

where $\alpha (,)$ is the Kuratowski's measure of non-compactness and $k$ is a number in $(0, 1)$ and independent of the bounded set $M$.

It is easy to see that any quasi-compact operator is a locally power $\alpha$-set contraction. (We recall that an element $R$ in $L(X)$ is said to be quasi-compact if the following properties hold:

1. $|| R^n || \leq K < \infty$, $n = 1, 2, 3, \ldots$

2. there exists an integer $m \geq 1$ and a compact element $Q$ in $L(X)$ such that

\[ || R^m - Q || < 1. \]

Let $X$ be as above and $T \in L(X)$ be a locally power $\alpha$-set contraction. Consider the equation

\[ x - Tx = y \]
where \( y \) is given. We call the Neumann series of \( T \) at \( y \) the (formal) series

\[
\sum_{n=0}^{\infty} T^n y.
\]

Then we have the following result.

**Theorem 3.** Let \( T \) be an element in \( L(X) \) be a locally power \( \gamma \)-set contraction. Then, in order that the Neumann series of \( T \) at \( y \) may be strongly convergent it is necessary and sufficient that

\[
\lim \| T^n y \| = 0.
\]

For the proof of this result we need some facts about linear locally power \( \alpha \)-set contractions which are given below as lemmas. For the proof we refer to the paper of G. Constantin (1) or the author's book (3).

**Lemma 4.** If \( S \in L(X) \) is a locally power \( \alpha \)-set contraction then

\[(z, x \in \sigma_p(S), |\sigma| \geq 1)\]

is a finite set. Here \( \sigma_p(\cdot, \cdot) \) is the point spectrum of \( (\cdot, \cdot) \).

**Lemma 5.** If \( S \in L(X) \) is a locally power \( \alpha \)-set contraction and \( z \) is a complex number with the following properties:

1) \[ |z| \geq 1, \]

2) \((y_n)\) is a sequence in \( X \) with the property that \((z-S)x_n = y_n, \lim y_n = y\)

where \((x_n)\) is a bounded sequence in \( X \).

Then the set \((x_n)\) is relatively compact and if \( \lim x_{n_k} = u \) then \( zu = Su = y \).

**Lemma 6.** If \( S \in L(X) \) is a locally power \( \alpha \)-set-contraction and \( z_0 \) is a complex number with \[ |z_0| \geq 1 \] and is not in \( \sigma_p(S) \) then \((z_0 - S)^{-1}\) (defined on the range of \((z_0 - S)\)) is a linear and continuous operator.

Using these results we prove the following assertion.

**Proposition 7.** If \( S \in L(X) \) is a locally power \( \alpha \)-set contraction then

\[ (z, |z| \geq 1) \cap \sigma_p(S) = \sigma(S) \cap (z, |z| \geq 1). \]

**Proof.** Let

\[ M = (z, z \in \sigma_p(S), |z| \geq 1) \]
and
\[ N = \{ z : \ | z | \geq 1 \} \setminus M. \]

If
\[ N_1 = N \cap \rho(S) \]
(\( \rho(S) \) is the resolvent set of \( S \)) then
\[ N = N_1 \cup N_2 \]
and the assertion of the proposition is proved if we show that \( N_2 \) is the empty set.

We remark that \( N \) is a connected set because, according to Lemma 4, \( M \) is a finite set. Also, \( N \) is an open set (in the relative topology).

Then, if \( N_2 \) is non-empty, we find \( z_0 \) in \( N_2 \) and a sequence \( (z_n) \) in \( N_1 \) such that
\[ \lim z_n = z_0. \]

But
\[ (\| (z_n - S)^{-1} \|) \]
is a bounded sequence and thus for some \( K > 0 \) we have
\[ \| (z_n - S)^{-1} \| \leq K. \]

We consider now the disc with the centre at \( z_0 \) and radius \( K^{-1} \), since \( z_n \in \rho(S) \) we have
\[ |z_0 - z_n| < K^{-1} \leq \| R(z_n, S) \|^{-1} \]
which gives that \( z_0 \) is a regular point for \( S \). This is a contradiction and thus \( N_2 \) is empty. The proposition is proved.

**Corollary 8.** Let \( S \in L(X) \) be a locally power \( \gamma \)-set contraction. Then \( (z, z \in \sigma(S), \ | z | \geq 1) \) and \( (z, z \in \sigma(S), \ | z | < 1) \) are spectral sets of \( S \) (i.e. these are closed and open subsets of \( \sigma(S) \)).

Now we are ready to prove Theorem 3.

We remark, as in (4), that we may suppose without loss of generality that \( X \) is the closed subspace generated by the subset \( \{ y, T y, T^2 y, \ldots \} \) and that the subset of all elements \( u \) in \( X \) with \( \lim T^a u = 0 \) is dense in \( X \).

Associated with the spectral sets \( (z, z \in \sigma(T), \ | z | \leq 1) \) and \( (z, z \in \sigma(T), \ | z | < 1) \) are the projections \( P \) and \( Q \) respectively. Since \( T \) is supposed to be
a locally power \( \alpha \)-set contraction it is easy to see that \( \mathcal{P}X \) is a finite dimensional subspace.

Now the proof of Theorem 3 can be continued exactly as in (4) and thus we omit the details.

REFERENCES