Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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On the convergence of Neumann series in Banach space.

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **76** (1984), n.1, p. 14–18.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1984_8_76_1_14_0>

Analisi matematica. — *On the convergence of Neumann series in Banach space.* Nota di VASILE I. ISTRĂŢESCU, presentata dal Corrisp. E. VESENTINI (*).

RIASSUNTO. — Si estende un risultato di N. Suzuki sulla convergenza della serie di Neumann per un operatore compatto in uno spazio di Banach.

0. INTRODUCTION

Let (S, B, μ) be a measure space with μ positive and finite. Consider the Fredholm equation

$$h(x) - \int_S K(x, y)\, h(y)\, d(y) = f(x)$$

where the kernel $K(\,,\,)$ is of Hilbert-Schmidt type. One of the oldest itera-ive methods for solving this equation is related to the so called Neumann series,

$$f(x) + \sum_{n=1}^{\infty} \int_S K_n(x, y) f(y)\, d\mu(y)$$

where $K_n(\,,\,)$ is the n-th kernel defined by $K(\,,\,)$. A sufficient condition for the convergence of the Neumann's series is that the spectral radius of the operator defined on $L^2(S, B, \mu)$ by

$$g(x) - \int_S K(x, y)\, g(y)\, d\mu(y)$$

be less than 1.

This may be considered as a special case of the following problem: Let X be a complex Banach space and T in L (X), L (X) is the set of all bounded

(*) Nella seduta del 14 gennaio 1984.

linear operators defined on X with values in X, and consider the equation

(1) $$x - \mathrm{T}x = y$$

with y given in X and T is supposed to be compact element in L (X).

A (iterative) scheme for solving the equation (1) is to consider the formal Neumann series

(2) $$\sum_{n=0} \mathrm{T}^n y .$$

It is obvious that if this is a convergent series then the sum gives the solution of the equation (1).

In (4) the following result is proved.

THEOREM 1. *Let* T *be in* L (X) *and compact. Then, in order that the Neumann series* (2) *may be strongly convergent it is necessary and sufficient that*

$$\lim \| \mathrm{T}^n y \| = 0 .$$

The purpose of the present Note is to show that there are other classes of operators in L (X) *for which a result like that in Theorem* 1 *is valid.*

In order to do this we recall the following definition of a class of (not necessarily linear) mappings.

DEPINITION 2. (2) *A continuous mapping* $f : \mathrm{X} \to \mathrm{X}$ *is said to be a locally power* α*-set contraction if for each non-compact bounded set* M *in* X *there exists an integer* $n = n(\mathrm{M})$ *such that*

$$\alpha(f^n(\mathrm{M})) \leqq k\alpha(\mathrm{M})$$

where $\alpha(,)$ *is the Kuratowski's measure of non-compactness and* k *is a number in* $(0 , 1)$ *and independent of the bounded set* M.

It is easy to see that any quasi-compact operator is a locally power α-set contraction. (We recall that an element R in L (X) is said to be quasi-compact if the following properties hold:

(1) $\| \mathrm{R}^n \| \leqq \mathrm{K} < \infty, \quad n = 1, 2, 3, \ldots,$

(2) there exists an integer $m \geqq 1$ and a compact element Q in L (X) such that

$$\| \mathrm{R}^m - \mathrm{Q} \| < 1 .$$

Let X be as above and $\mathrm{T} \in \mathrm{L}(\mathrm{X})$ be a locally power α-set contraction. Consider the equation

$$x - \mathrm{T}x = y$$

where y is given. We call the Neumann series of T at y the (formal) series

$$\sum_{n=0}^{\infty} T^n y .$$

Then we have the following result.

THEOREM 3. *Let* T *be an element in* L (X) *be a locally power* γ*-set contraction. Then, in order that the Neumann series of* T *at* y *may be strongly convergent it is necessary and sufficient that*

$$\lim \| T^n y \| = 0 . \tag{3}$$

For the proof of this result we need some facts about linear locally power α-set contractions which are given below as lemmas. For the proof we refer to the paper of G. Constantin (1) or the author's book (3).

LEMMA 4. *If* $S \in L(X)$ *is a locally power* α*-set contraction then*

$$(z \,.\, z \in \sigma_p(S) \,,\ | \sigma | \geqq 1)$$

is a finite set. Here $\sigma_p(\,,\,)$ *is the point spectrum of* $(\,,\,)$.

LEMMA 5. *If* $S \in L(X)$ *is a locally power* α*-set contraction and* z *is a complex number with the following properties:*

1) $| z | \geqq 1$,

2) (y_n) is a sequence in X with the property that $(z - S) x_n = y_n$, $\lim y_n = y$

where (x_n) *is a bounded sequence in* X.

Then the set (x_n) is relatively compact and if $\lim x_{n_k} = u$ then $zu - Su = y$.

LEMMA 6. *If* $S \in L(X)$ *is a locally power* α*-set-contraction and* z_0 *is a complex number with* $| z_0 | \geqq 1$ *and is not in* $\sigma_p(S)$ *then* $(z_0 - S)^{-1}$ *(defined on the range of* $(z_0 - S)$*) is a linear and continuous operator.*

Using these results we prove the following assertion.

PROPOSITION 7. *If* $S \in L(X)$ *is a locally power* α*-set contraction then*

$$(z \,,\ | z | \geqq 1) \cap \sigma_p(S) = \sigma(S) \cap (z \,,\ | z | \geqq 1) .$$

Proof. Let

$$M = (z \,,\ z \in \sigma_p(S) \,,\ | z | \geqq 1)$$

and

$$N = (z \,,\, |z| \geqq 1) \setminus M \,.$$

If

$$N_1 = N \cap \rho(S)$$

($\rho(S)$ is the resolvent set of S) then

$$N = N_1 \cup N_2$$

and the assertion of the proposition is proved if we show that N_2 is the empty set.

We remark that N is a connected set because, according to Lemma 4, M is a finite set. Also, N is an open set (in the relative topology).

Then, if N_2 is non-empty, we find z_0 in N_2 and a sequence (z_n) in N_1 such that

$$\lim z_n = z_0 \,.$$

But

$$(\| (z_n - S)^{-1} \|)$$

is a bounded sequence and thus for some $K > 0$ we have

$$\| (z_n - S)^{-1} \| \leqq K \,.$$

We consider now the disc with the centre at z_0 and radius K^{-1}, since $z_n \in \rho(S)$ we have

$$| z_0 - z_n | < K^{-1} \leqq \| (R(z_n, S)) \|^{-1}$$

which gives that z_0 is a regular point for S. This is a contradiction and thus N_2 is empty. The proposition is proved.

COROLLARI 8. *Let* $S \in L(X)$ *be a locally power* γ*-set contraction. Then* $(z \,,\, z \in \sigma(S) \,,\, |z| \geqq 1)$ *and* $(z \,,\, z \in \sigma(S) \,,\, |z| < 1)$ *are spectral sets of* S (*i.e. these are closed and open subsets of* $\sigma(S)$).

Now we are ready to prove Theorem 3.

We remark, as in (4), that we may suppose without loss of generality that X is the closed subspace generated by the subset $(y, Ty, T^2 y, \ldots)$ and that the subset of all elements u in X with $\lim T^n u = 0$ is dense in X.

Associated with the spectral sets $(z \,,\, z \in \sigma(T) \,,\, |z| \leqq 1)$ and $(z \,,\, z \in \sigma(T), |z| < 1)$ are the projections P and Q respectively. Since T is supposed to be a locally power α-set contraction it is easy to see that PX is a finite dimensional subspace.

Now the proof of Theorem 3 can be continued exactly as in (4) and thus we omit the details.

References

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