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A classification for real and complex finite dimensional $\mathcal{F}^*$-algebras

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Geometria. — *A classification for real and complex finite dimensional J*-algebras.* Nota di MAURO MESCHIARI (*), presentata (**) dal Corrisp. E. VESENTINI.

RIASSUNTO. — La presente Nota contiene una lista di J*–algebre reali di dimensione finita ed una lista di J*–algebre complesse di dimensione finita tali che: 1) due elementi distinti di ogni lista non sono mai J*–isomorfi; 2) ogni J*–algebra di dimensione finita reale (complessa) è J*–isomorfa su **R** (su **C**) alla somma diretta, finita, di J*–algebre reali (complesse) elencate nella lista. In altre parole, diamo qui una classificazione completa delle J*–algebre reali e delle J*–algebre complesse di dimensione finita. Nel caso complesso, la nostra classificazione coincide con quella data (per la dimensione finita) da L. A. Harris in [2] ove si elencano quattro classi infinite di J*–algebre corrispondenti ai quattro tipi di spazi di matrici associati alla classificazione di E. Cartan dei domini limitati simmetrici irriducibili. Una immediata conseguenza della nostra classificazione è la non esistenza di J*–algebre complesse il cui disco unitario sia uno dei due domini eccezionali della classificazione di E. Cartan, un risultato già ottenuto da O. Loos e K. McCrimmon in [3].

The present Note contains a list of real finite dimensional J*-algebras and a list of complex finite dimensional J*-algebras with the following properties:

1) different elements of the list are not J*-isomorphic;
2) every finite dimensional real (complex) J*-algebra is J*-isomorphic over **R** (over **C**) to a direct sum or real (complex) J*-algebras of the list.

In other words, we give a complete classification of real and complex finite dimensional J*-algebras. Our classification, for finite dimensional complex J*-algebras coincides with the one given by L.A. Harris (see [2]) listing four infinite classes corresponding to the four types of matrix spaces associated with E. Cartan's classification of irreducible bounded symmetric domains. As a consequence of our classification, it turns out that there is no complex J*-algebra whose unity ball is any one of the two exceptional domains in E. Cartan's classification; a result previously obtained by O. Loos and K. McCrimmon in [3].

Complete proofs and further details will appear elsewhere.

(*) Istituto Matematico «G. Vitali», Università, Via G. Campi 213/B, 41100 Modena.

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1. Definitions

Let **K** be either the real or complex field and let V and W be two Hilbert vector spaces on **K**. Moreover let $L_K(V, W)$ denote the Banach vector space of all bounded **K**-linear operators from V into W (endowed with the operator norm) and let the same symbol $\langle\,,\,\rangle_K$ denote the scalar product both in V and in W.

A **K**-J*-algebra j of **K**-linear operators from V into W is a closed **K**-vector subspace of $L_K(V, W)$ such that $aa^* a \in j$ whenever $a \in j$ ($a^*: W \to V$ is here the adjoint operator of a).

Since a Hilbert space on **C** is also naturally a Hilbert space on **R**, for the scalar product $\langle x, y\rangle_R = \frac{1}{2}(\langle x, y\rangle_C + \langle y, x\rangle_C)$, any **C**-J*-algebra has a natural structure of **R**-J*-algebra.

Let j be a **K**-J*-algebra. Setting

$$\{a\,b\,c\} = \frac{1}{6}(a\,b^*\,c + c\,b^*\,a + a\,c^*\,b + b\,c^*\,a + b\,a^*\,c + c\,a^*\,b)$$

it is easily checked that $\{a\,b\,c\} \in j$ whenever a, b, c are in j.

Two **K**-J*-algebras j and j' are said to be **K**-J*-isomorphic if there exists a **K**-linear bijection $L: j \to j'$ such that $L(a\,a^*\,a) = L(a)\,L(a)^*\,L(a)$ whenever $a \in j$.

Let $L: j \to j'$ be a **K**-J*-isomorphism, we have:

$$L(\{a\,b\,c\}) = \{L(a)\,L(b)\,L(c)\} \text{ whenever } a, b, c \in j;$$

$$|L(a)| = |a| \text{ whenever } a \in j.$$

In the following we shall frequently use the observation:

Two **R**-J**-algebras j and j' are* **R**-J**-isomorphic if, and only if, there exists a* **R**-*base* B *of* j *and a mapping* $L: B \to j'$ *such that*:

$L(B)$ *is a* **R**-*base of* j';

$L(\{a\,b\,c\}) = \{L(a)\,L(b)\,L(c)\}$ *whenever* $a, b, c \in j$.

Now we give some more definitions that are preserved by **R**-J*-isomorphisms, and hence by **C**-J*-isomorphisms.

An element a of a J*-algebra j is said to be *idempotent* if $\{a\,a\,a\} = a$; *irreducible* if $3\{a\,a\{a\,a\,b\}\} - \{a\,a\,b\}$ is a scalar multiple of a for all $b \in j$.

For any idempotent element $a \in j$, the **K**-linear mapping $\psi_a: j \to j$ defined by $\psi_a(x) = 4\{a\,a\,x\} - 3\{a\,a\,x\}\}$, is a projection on j.

Two idempotent elements $a, b \in j$ are said to be *strongly independent* if $\psi_a(b) = \psi_b(a) = 0$. If a and b are two non-zero, strongly independent, idempotent elements of j, then a and b are linearly independent too.

Let j be a finite dimensional J*-algebra. We define the *height* of j, Hght (j), as the maximum of the cardinality of the subsets of j whose elements are non-zero, idempotent and mutually strongly independent.

A J*-algebra j is said to be *irreducible* if either Hght $(j)=1$ or Hght $(j) \geq 2$ and whenever a and b are two non-zero, strongly independent, idempotent elements of j, $\psi_a \psi_b (j)$ is different from $\{0\}$.

Let j and j' be two **K**-J*-algebras of bounded operators of the Hilbert space V and W into the Hilbert spaces V′ and W′ respectively. The set $j \times j'$ has a natural structure of **K**-J*-algebra of linear operators of $V \oplus V'$ into $W \oplus W'$. Such an algebra is the *direct sum* of j and j'.

Any finite dimensional **K**-J*-algebra is **K**-J*-isomorphic to a direct sum of a finite number of finite dimensional irreducible **K**-J*-algebras. This decomposition turns out to be essentially unique. Hence we have a complete classification of finite dimensional **K**-J*-algebras simply by classifying the irreducible ones.

2. SOME EXAMPLES OF MATRIX J*-ALGEBRAS

Denote by **F** either **R**, **C** or the non-commutative field of Hamilton's quaternions, **H**, and let $M(m, n; \mathbf{F})$ denote the set of $m \times n$ matrices with entries in F. Identifying $\mathbf{C}^s$ with $\mathbf{R}^{2s}$ and $\mathbf{H}^s$ with $\mathbf{R}^{4s}$, $M(m, n; F)$ becomes a space of bounded **R**-linear of the Hilbert space, on **R**, $\mathbf{F}^n$ into the Hilbert space, on **R**, $\mathbf{F}^m$.

A **R**-subspace of $M(m, n; \mathbf{F})$ is a *matrix* **R**-*J*-algebra* if the corresponding space of operators is. The same fact holds for *matrix* **C**-*J*-algebras*.

Let $A \in M(m, n; F)$. ${}^tA \in M(n, m; \mathbf{F})$ denotes the transposed matrix of A; $\bar{A} \in M(m, n; F)$ denotes the conjugate matrix of A (the matrix whose entries are the conjugate, in F, of the corresponding entries of A). The adjoint of A is given by $A^* = {}^t\bar{A}$.

Now we list some families of matrix **R**-J*-algebras:

I) $M(m, n; \mathbf{R})$, $\quad n \geq m \geq 1$;

II) $M(m, n; \mathbf{C})$, $\quad n \geq m \geq 1$;

III) $M(m, n; \mathbf{H})$, $\quad n \geq m \geq 1$;

IV) $A(n; \mathbf{R}) = \{A : A \in M(n, n; \mathbf{R})$ with $A + {}^tA = 0\}$, $\quad n \geq 2$;

V) $A(n, \mathbf{C}) = \{A : A \in M(n, n; \mathbf{C})$ with $A + {}^tA = 0\}$, $\quad n \geq 2$;

VI) $S(n, \mathbf{R}) = \{A : A \in M(n, n; \mathbf{R})$ with $A - {}^tA = 0\}$, $\quad n \geq 2$;

VII) $S(n, \mathbf{C}) = \{A : A \in M(n, n; \mathbf{C})$ with $A - {}^tA = 0\}$, $\quad n \geq 2$;

VIII) $H(n, \mathbf{C}) = \{A : A \in M(n, n; \mathbf{C})$ with $A - A^* = 0\}$, $\quad n \geq 2$;

IX) $H(n, \mathbf{H}) = \{A : A \in M(n, n; \mathbf{H})$ with $A - A^* = 0\}$, $\quad n \geq 2$;

X) $H^-(n, \mathbf{H}) = \{A : A \in M(n, n; \mathbf{H})$ with $A + A^* = 0\}$, $\quad n \geq 2$.

Note that the matrix **R**-J*-algebras II), V) and VII) are **C**-J*-algebras too.

We now construct other matrix **R**-J*-algebras useful to our classification.

For any $n \geq 1$, let us define three **R**-linear mappings, Φ, Λ, $\Omega : M(n, n; \mathbf{R}) \to M(2n, 2n; R)$ by:

1) $\Omega(\| a_{ij} \|) = \| b_{rs} \|$,
   where $b_{2i\,2j} = b_{2i-1\,2j-1} = a_{ij}$ and $b_{2i\,2j-1} = b_{2i-1\,2j} = 0$ for all i, $j = 1, \cdots, n$;

2) $\Lambda(A) = \begin{Vmatrix} 0 & A \\ A & 0 \end{Vmatrix}$, whenever $A \in M(n, n; \mathbf{R})$;

3) $\Phi(A) = \begin{Vmatrix} A & 0 \\ 0 & -A \end{Vmatrix}$, whenever $A \in M(n, n; \mathbf{R})$.

The three mappings Φ, Λ and Ω satisfy the following identities:

i) $\Phi\Omega = \Omega\Phi$;

ii) $\Lambda\Omega = \Omega\Lambda$;

iii) $\Omega(AB) = \Omega(A)\,\Omega(B)$, whenever $A, B \in M(n, n; \mathbf{R})$;

iv) $\Omega(A)^* = \Omega(A^*)$, whenever $A \in M(n, n; \mathbf{R})$;

v) $\Phi(A)^* = \Phi(A^*)$, whenever $A \in M(n, n; \mathbf{R})$;

vi) $\Lambda(A)^* = \Lambda(A^*)$, whenever $A \in M(n, n; \mathbf{R})$.

Let:

$U_0 = \varnothing$ (the empty set);

$U_n = \{\Omega^{n-1}(\| 1 \|)\} \cup \left\{\Omega^{i-2}\,\Phi^{n-i}\left(\begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}\right) : 2 \leq i \leq n\right\}$, for any $n \geq 1$;

and let

$$S = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{Vmatrix}$$

Now, we define:

$T_{2n} = \{\Omega^{i-1}\,\Lambda^{n-i-1}(S) : 1 \leq i \leq n-1\} \cup \{\Omega^{i-1}\,\Lambda^{n-i-1}({}^tS) : 1 \leq i \leq n-1\} \cup \left\{\Omega^{n-1}\left(\begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix}\right), \Omega^{n-1}\left(\begin{Vmatrix} 0 & 0 \\ 0 & -1 \end{Vmatrix}\right)\right\}$, for any $n \geq 1$;

$T_{2n,m} = \Omega^{n-1}(T_{2n}) \cup \Lambda^n\left(\left\{\begin{Vmatrix} 0 & R \\ {}^tR & 0 \end{Vmatrix} : R \in U_m\right\}\right)$, for any $m \geq 0$ and $n \geq 1$;

$E_n = \left\{\Omega^{n-1}\,\Lambda^{i-1}\left(\begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}\right) : 1 \leq i \leq n\right\}$, for any $n \geq 1$.

Whenever M is any subset of $M(m, n; \mathbf{F})$, let $V_{\mathbf{R}}(M)$ be the **R**-vector subspace of $M(m, n; \mathbf{F})$ with M as a set of generators (a **C**-vector space $V_{\mathbf{C}}(M)$ is defined similarly, when possible). Using the identities i), ..., vi), it is easy to prove that:

$$\{A\,B\,C\} \in V_{\mathbf{R}}(T_{2n,m}) \qquad \text{whenever} \quad A, B, C \in T_{2n,m};$$

$$A\,{}^tBC + C\,{}^tBA \in V_{\mathbf{R}}(E_n) \qquad \text{whenever} \quad A, B, C \in E_n.$$

The $V_{\mathbf{R}}(T_{2n,m})$ turns out to be a **R**-J*-algebra and $V_{\mathbf{C}}(E_n)$ a **C**-J*-algebra. Let us denote:

XI) $\mathrm{T}_{2n,m} = V_{\mathbf{R}}(T_{2n,m})$;

XII) $\mathrm{E}_n = V_{\mathbf{C}}(E_n)$.

3. A CLASSIFICATION FOR FINITE DIMENSIONAL R-J*-ALGEBRAS

Let j be a **R**-J*-algebra of height 1 and dimension $n \geq 1$. j has a basis B, as a real vector space, with the following *structure relations*:

$\{a\,a\,a\} = a$, for all $a \in B$;

$\{a\,a\,b\} = \dfrac{1}{3}\,b$, for all a, b in B with $a \neq b$;

$\{a\,b\,c\} = 0$, whenever a, b, c are three distinct elements of B.

Since the natural basis, B′, of the spaces of row matrices, $M(1, n; \mathbf{R})$, has the above structure relations too, then, any bijection $L: B \to B'$ can be extended to a **R**-J*-isomorphism $L: j \to M(1, n; \mathbf{R})$.

Assume, now, that the **R**-J*-algebra j is irreducible and has height $h \geq 2$. Whenever a and b are two non-zero, strongly independent, irreducible, idempotent elements of j, the sets:

$\delta_a(j) = \{aa^*\,xa^*\,a : x \in j)$;

$\delta_b(j) = \{bb^*\,xb^*\,b : x \in j\}$;

$\psi_a\,\psi_b(j)$;

and

$$j_{ab} = \delta_a(j) + \delta_b(j) + \psi_a\,\psi_b(j)$$

are **R**-J*-subalgebras of j. j_{ab} results **R**-J*-isomorphic to one of the following **R**-J*-algebras: $\mathrm{T}_{2n,m}$, $S(2; \mathbf{C})$, $H^-(2, \mathbf{H})$, $M(2, 2; \mathbf{H})$, E_{n+2} ($m \geq 0$, $n \geq 1$ and $(m, n) \neq (0, 1)$).

Distinguishing various particular cases according to the height of j, the dimension of $\delta_a(j)$ and the existence, in $\psi_a \psi_b(j)$, of special subsets of non-zero, irreducible, idempotent elements of j, we obtain the following

THEOREM 1. *Any irreducible finite dimensional* **R**-J*-*algebra is* **R**-J*-*isomorphic to exactly one of the following irreducible matrix* **R**-J*-*algebras*:

I) $M(m, n; \mathbf{R})$, $n \geq m \geq 1$;

II) $M(m, n; \mathbf{C})$, $n \geq m \geq 2$;

III) $M(m, n; \mathbf{H})$, $n \geq m \geq 2$;

IV) $A(n, \mathbf{R})$, $n \geq 4$;

V) $A(n; \mathbf{C})$, $n \geq 4$;

VI) $S(n; \mathbf{R})$, $n \geq 3$;

VII) $S(n; \mathbf{C})$, $n \geq 2$;

VIII) $H(n; \mathbf{C})$, $n \geq 3$;

IX) $H(n; \mathbf{H})$, $n \geq 3$;

X) $H^-(n; \mathbf{H})$, $n \geq 2$;

XI) $T_{2n,m}$, $n \geq 1$, $m \geq 0$ and $(m, n) \neq (0, 1), (0, 2)$;

XII) E_n, $n \geq 5$.

4. A classification for finite dimensional complex J*-algebras

Let j be a **C**-J*-algebra of bounded **C**-linear operators from the complex Hilbert space V into the complex Hilbert space W. Let us denote by J both almost complex structures operating on V and W. Whenever $a \in j$, we have:

$$Ja = aJ \in j;$$
$$(Ja)^* = -Ja^*.$$

Using the above relations we prove that whenever a is a non-zero, irreducible idempotent element of j, $\{Ja, a\}$ is a **R**-base for $\delta_a(j)$. Now we distinguish two cases according to the height of j.

If the height of j is 1, then j is **C**-J*-isomorphic to $M(1, n; \mathbf{C})$ ($n = \dim_{\mathbf{C}} j$).

If the height of j is greater than 1, and j is irreducible, Theorem 1 implies that j is **R**-J*-isomorphic only to one of the **R**-J*-algebras of type II), V), VII) and XII), proving thereby

LEMMA. *Let j be a non-zero, irreducible* **C**-J**-algebra. j is* **R**-J**-isomorphic to one of the following* **C**-J**-algebras*:

A) $M(m, n; \mathbf{C})$, $n \geq m \geq 1$;

B) $A(n; \mathbf{C})$, $n \geq 4$;

C) $S(n; \mathbf{C})$, $n \geq 2$;

D) E_n , $n \geq 5$.

Let H be one of the **C**-J*-algebras listed in the above lemma, and let $L: H \to H$ be the **R**-J*-isomorphism defined by $L(x) = \bar{x}$ (the conjugate matrix of x). Clearly, if $F: j \to H$ is a **R**-J*-isomorphism, LF is too. Now it can be proved that either F or LF is a **C**-J*-isomorphism.

THEOREM 2. *Any irreducible finite dimensional complex* J**-algebra is* **C**-J**-isomorphic to one of the following matrix* **C**-J**-algebras*:

A) $M(m, n; \mathbf{C})$, $n \geq m \geq 1$;

B) $A(n; \mathbf{C})$, $n \geq 4$;

C) $S(n; \mathbf{C})$, $n \geq 2$;

D) E_n , $n \geq 5$.

In [2] L. A. Harris investigated the bounded domains $D(j) = \{x : x \in j$ with $|x| \leq 1\}$ obtained in term of a complex J*-algebra j, and showed that all of them are bounded symmetric domains. Hence he suggested that E. Cartan's classification for bounded symmetric domains of finite dimensions, [1], could yield a classification of finite dimensional complex J*-algebras.

Moreover L.A. Harris listed four infinite classes of complex J*-algebras whose unity balls correspond to one of E. Cartan's domains of type I), . . . , IV) (our classes A), . . . , D)).

The question whether there existed any complex J*-algebra j such that $D(j)$ be one of the two exceptional E. Cartan's domains in dimension 16 and 27, remained open until O. Loos and K. McCrimmon gave a negative answer in [3]. Our theorem 2 gives a direct classification of finite dimensional complex J*-algebras and solves, independently, the above question.

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