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On the Stefan problem with energy specification

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Equazioni a derivate parziali. — *On the Stefan problem with energy specification.* Nota (*) di PIERLUIGI COLLI, presentata (**) dal Corrisp. E. MAGENES.

RIASSUNTO. — Vengono trattati due problemi di Stefan con la specificazione dell'energia. Dapprima si fornisce una formulazione debole di un problema unidimensionale ad una fase studiato in [4]: si dimostra un risultato di esistenza. In seguito si considera un problema di Stefan pluridimensionale e multifase in cui viene assegnata la energia totale del sistema ad ogni istante; si mostra l'esistenza e l'unicità della soluzione per due formulazioni provando inoltre l'equivalenza fra queste.

0. INTRODUCTION

In this work we study a Stefan problem with an unusual boundary condition which aims to represent an energy specification. We start from the consideration of two papers by Cannon and van der Hoek [4, 5], which deal with two one-dimensional Stefan problems of one phase and two phases respectively, with the following condition:

$$\int_0^{s(t)} \theta(x, t)\, dx = E(t) \tag{0.1}$$

where θ is the temperature of the system, $s(t)$ is the free boundary, $E(t)$ is given. In both cases they proved existence and uniqueness results, exploiting typical methods of the one-dimensional case.

Here, in section 1, we study the one-phase problem in a weak formulation, transforming the unknown θ by time and space integrations, following Baiocchi [1] and Duvant [8]. We introduce the "freezing index" $w(x, t) = \int_0^t \theta(x, \tau)\, d\tau$ and also the function $z(x, t) = \int_x^c w(\xi, t)\, d\xi$. Now z satisfies a non-linear equation, for which we prove the existence of at least one solution.

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In section 2 we observe that (0.1) does not represent the energy of the system, because in the Stefan problem the total energy (more precisely, the total enthalpy) is given by the integral of the enthalpy density. Therefore, we write a new problem with the condition $\int_0^c u(x,t)\,\mathrm{d}x = \mathrm{E}(t)$, where u represents the enthalpy density. We study this problem in more space variables in the unknowns u and w. We prove existence and uniqueness of the solution u in a first weak formulation, using general results of [6]. Then we show the equivalence between this formulation and that one in terms of the freezing index, which leads to a variational inequality.

1. Weak formulation with temperature integral

We study the one-phase one-dimensional Stefan problem with the condition considered in [4]. Given $\mathrm{T} > 0$, $b > 0$, $\mathrm{E}(t)$ in $[0,\mathrm{T}]$, $\phi(x)$ in $[0,b]$, we look for two functions $s:[0,\mathrm{T}] \to (0,+\infty)$ with $s(0) = b$, and $u : \Omega \to \mathbb{R}$ where $\Omega = \{(x,t) : 0 < x < s(t),\ 0 < t < \mathrm{T}\}$ such that:

$$\theta_t - \theta_{xx} = 0 \qquad \text{in } \Omega \tag{1.1}$$

$$\theta(x,0) = \phi(x) \qquad \text{for } 0 \le x \le b \tag{1.2}$$

$$\int_0^{s(t)} \theta(x,t)\,\mathrm{d}x = \mathrm{E}(t) \qquad \text{for } 0 < t \le \mathrm{T} \tag{1.3}$$

$$\theta(s(t),t) = 0\,, \quad \theta_x(s(t),t) = -s'(t) \qquad \text{for } 0 < t \le \mathrm{T}\,. \tag{1.4}$$

Cannon-van der Hoek prove existence, uniqueness and positiveness of the classical solution of this problem by using methods which are typical of the one-dimensional case. We are interested in a weak formulation of the problem, in order to weaken the hypotheses of [4] on the data and to work also in the multi-dimensional case. Let $c > 0$ be a constant such that $0 < s(t) < c$ for $0 \le t \le \mathrm{T}$. If $\mathrm{Q} = (0,c) \times (0,\mathrm{T})$, then $\Omega \subset \mathrm{Q}$. We denote by $\tilde{\theta}$ the extension of θ to Q with zero value and we introduce, as in [8], the freezing index:

$$w(x,t) = \int_0^t \tilde{\theta}(x,\tau)\,\mathrm{d}\tau \qquad (x,t) \in \mathrm{Q}\,. \tag{1.5}$$

It is well-known that, if $\{s,\theta\}$ is the classical solution of the problem and w is defined by (1.5), then we have (see [12] for the details):

$$w_t - w_{xx} + \chi_\Omega = f \qquad \text{in the sense of } \mathscr{D}'(\mathrm{Q}) \tag{1.6}$$

$$\int_0^c w(x,t)\,\mathrm{d}x = \mathrm{G}(t) \qquad \text{for } 0<t<\mathrm{T} \tag{1.7}$$

$$w_x(c,t)=0 \qquad \text{for } 0<t<\mathrm{T} \tag{1.8}$$

$$w(x,0)=0 \quad \text{for } 0<x<c, \quad w>0 \quad \text{in } \Omega, \qquad w=0 \quad \text{in } \mathrm{Q}-\Omega \tag{1.9}$$

where χ_Ω is the characteristic function of Ω, $f(x)=1+\phi(x)$ if $0<x<b$, $f(x)=0$ if $b<x<c$, $\mathrm{G}(t)=\int_0^t \mathrm{E}(\tau)\,\mathrm{d}\tau$. If H denotes the Heaviside function ($\mathrm{H}(x)=\{1\}$ if $x>0$, $\mathrm{H}(x)=[0,1]$ if $x=0$, $\mathrm{H}(x)=\{0\}$ if $x<0$), then $\chi_\Omega \in \mathrm{H}(w)$. Therefore it is possible to write a new problem in the unknown w, which must satisfy $w_t - w_{xx} - f \in -\mathrm{H}(w)$ in the sense of $\mathscr{D}'(\mathrm{Q})$ and the boundary conditions in (1.7)-(1.9). But it seems difficult to study this problem, as the condition (1.7) is non-local. We introduce a new unknown function:

$$z(x,t)=\int_x^c w(\xi,t)\,\mathrm{d}\xi \qquad (x,t)\in \mathrm{Q} \tag{1.10}$$

which is such that $z>0$ where $w>0$, $z=0$ where $w=0$, since the free boundary can be expressed as a function of t in x. Set $\mathrm{F}(x)=\int_x^c f(\xi)\,\mathrm{d}\xi$. If $\{s,\theta\}$ is the classical solution of the problem and w and z are defined by (1.5) and (1.10), we have:

$$z_t - z_{xx} = -\int_x^c \chi_\Omega\,\mathrm{d}\xi + \mathrm{F}(x) \qquad \text{in the sense of } \mathscr{D}'(\mathrm{Q}) \tag{1.11}$$

$$z(x,0)=0 \qquad 0\le x\le c \tag{1.12}$$

$$z(0,t)=\mathrm{G}(t) \qquad 0<t<\mathrm{T} \tag{1.13}$$

$$z(c,t)=0 \qquad 0<t<\mathrm{T}. \tag{1.14}$$

We integrate (1.6) from x to c to obtain (1.11). Notice that $\chi_\Omega \in \mathrm{H}(\tilde\theta) = \mathrm{H}(w) = \mathrm{H}(z)$; starting from this, we consider a new problem in the function z, which we write specifying the spaces of the data and the unknowns:

(Pb. 1) given $F \in L^2(0, c)$ and $G \in H^{3/4}(0, T)$, find two functions $z \in H^{2,1}(Q)$ and $g \in L^\infty(Q)$ such that

$$z_t - z_{xx} = -\int_x^c g(\xi, t)\, d\xi + F(x) \qquad \text{a.e. in Q} \tag{1.15}$$

$$g \in H(z) \tag{1.16}$$

and such that z satisfies (1.12)-(1.14).

THEOREM 1. *There exists at least one solution of* (Pb. 1).

Proof. We consider an approximation H_n of H ($H_n(x) = H(x)$ if $x < 0$ or $x > 1/n$; $H_n(x) = nx$ if $0 \le x \le 1/n$) which is Lipschitz-continuous of constant n. We approximate (Pb. 1) by:

(Pb. 2) given F and G as in (Pb. 1), for any fixed $n \in \mathbf{N}$ find $z_n \in H^{2,1}(Q)$ such that z_n satisfies (1.12)-(1.14) (where z is replaced by z_n) and such that:

$$\frac{\partial z_n}{\partial t} - \frac{\partial^2 z_n}{\partial x^2} = -\int_x^c H_n(z_n)\, d\xi + F(x) \qquad \text{a.e. in Q.} \tag{1.17}$$

LEMMA. *There exists one and only one solution of* (Pb. 2).

Proof. We fix n and let μ be a function in $L^2(Q)$; we consider the operator $\Psi: L^2(Q) \to H^{2,1}(Q)$, $\Psi(\mu) = v$, v verifying (1.12)-(1.14) with z replaced by v, and such that $v_t - v_{xx} = -\int_x^c H_n(\mu)\, d\xi + F(x)$ a.e. in Q. The problem of finding v is well-posed (see, for example, [11]), i.e. Ψ is continuous withr espect to the strong topology.

Existence. v is bounded in $H^{2,1}(Q)$ and then Ψ can be restricted to a weakly compact convex set in $H^{2,1}(Q)$. By the Schauder Theorem, there exists at least one fixed point for Ψ.

Uniqueness. Let $\tilde{z}_1, \tilde{z}_2 \in L^2(Q)$, $z_i = \Psi(\tilde{z}_i)$, $i = 1, 2$. We set $z = z_1 - z_2$, $\tilde{z} = \tilde{z}_1 - \tilde{z}_2$. Notice that $z_t - z_{xx} = -\int_x^c \{H_n(\tilde{z}_1) - H_n(\tilde{z}_2)\}$ a.e.

in Q and $z(0, t) = 0$ for $t \in (0, T)$. We use z as test function and we integrate in x getting:

$$(1.18) \qquad \frac{1}{2}\frac{d}{dt}\left\{\int_0^c |z(x,t)|^2\, dx\right\} + \int_0^c |z_x(x,t)|^2\, dx =$$

$$= -\int_0^c z(x,t)\int_x^c \{H_n(\tilde{z}_1) - H_n(\tilde{z}_2)\}\, d\xi\, dx .$$

Now we integrate (1.18) from 0 to t, we bound the second member using the fact that H_n is Lipschitz-continuous, then we take the maximum between 0 and $\bar{t}\,(>0)$. It follows that Ψ is a contraction mapping in $C^0([0, \bar{t}]; L^2(0, c)) \cap L^2(0, \bar{t}; H^1_0(0, c))$ if $\bar{t} < M$, where M depends only on c and n; therefore it is possible to iterate the scheme. The fixed point is unique.

By the Lemma, we have the existence of a sequence of solutions of (Pb. 2). As z_n is bounded in $H^{2,1}(Q)$ and $H_n(z_n)$ in $L^\infty(Q)$, there exist z and g such that, possibly taking subsequences, $z_n \to z$ weakly in $H^{2,1}(Q)$ and $H_n(z_n) \to g$ weakly star in $L^\infty(Q)$. Taking the limit, as an immediate consequence, we obtain that z satisfies (1.12)-(1.14). In order to take the limit in (1.17), we must prove that $\int_x^c H_n(z_n)\, d\xi \to \int_x^c g\, d\xi$ weakly in $L^2(Q)$. For this end, it is enough to apply Fubini's Theorem twice in

$$\left(\int_x^c H_n(z_n)\, d\xi, v\right)_{L^2(Q)} = \iint_Q \int_0^c H_n(z_n)\, \chi_{(x,c)}\, v\, d\xi\, dx\, dt, \qquad v \in L^2(Q)$$

to get

$$\left(H_n(z_n), \int_0^c \chi_{(x,c)} \cdot v\, dx\right)_{L^2(Q)} \to \left(g, \int_0^c \chi_{(x,c)}\, v\, dx\right)_{L^2(Q)} .$$

It remains to prove (1.16). We have $H_n = \partial P_n$, $H = \partial P$, where P_n and P are the primitives of H_n and H which vanish at $x = 0$. Besides $P_n \to P$ uniformly when $n \to \infty$. Being $H_n = \partial P_n$, we get

$$(1.19) \qquad P_n(z_n) - P_n(v) \leq (H_n(z_n), z_n - v)_{L^2(Q)} \qquad \forall v \in H^{2,1}(Q) .$$

We obtain, taking the limit for $n \to \infty$ in (1.19)

$$(1.20) \qquad P(z) - P(v) \leq (g, z - v)_{L^2(Q)} \qquad \forall v \in H^{2,1}(Q)$$

that is $g \in H(z)$.

Remark 1. The uniqueness of the solution of (Pb. 1) is an open question. Another open question is the possibility of getting a classical solution by a solution of (Pb. 1), if the data G and F are more regular (the assumptions on G and F are very weak compared with those of [4]). For instance, an important property to recover for z is the positiveness: notice that θ is positive on Ω and also z must be so.

Remark 2. We could also study other weak formulations of the problem, for example that in w or that of an analogous problem where (1.8) is replaced by $w(c, t) = 0$. In this case it is important to find the relations between this formulation and the classical one.

2. Weak formulation with enthalpy integral

In [4] the authors claim that the condition (1.3) is a specification of the energy. As a matter of fact at any time the total energy is given by the space integral of the enthalpy density. In several works, as [2, 3, 7, 9, 10], boundary value heat problems are studied with conditions similar to (1.3); in these cases, the temperature integral has the physical meaning of energy, if the temperature is assumed proportional to the enthalpy. If now we take a free boundary problem, the enthalpy density plays the rôle of unknown of the system. Note that the enthalpy determines the temperature, but the converse is not true. Now we consider a multi-phase problem (which is more general physically) and we assume the enthalpy integral as a datum; furthermore we generalize to the more space variables case. The question arises of how many and which boundary conditions are to be specified, if at any time we want to consider also the total energy as a datum. We give a Dirichlet condition on the fixed boundary determined up to a constant, which will be specified precisely by the energy specification.

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with sufficiently smooth boundary Γ of class C^2, for instance; Ω is locally on only one side of Γ. We use the notations $Q = \Omega \times (0, T)$, $\Sigma = \Gamma \times (0, T)$ as in [13], to which we shall refer constantly in the sequel. We denote by u the enthalpy for unitary mass; now u and the temperature θ are related by $\theta = \beta(u)$, with $\beta(u) = u - \lambda$ if $u > \lambda$, $\beta(u) = 0$ if $0 \le u \le \lambda$, $\beta(u) = u$ if $u < 0$, where λ is the latent heat. Note that β is Lipschitz-continuous and denote by j the primitive of β which vanishes at 0. We write the following formal prrblem:

(Pb. 3) given $u_0 : \Omega \to \mathbf{R}$, $E : (0, T) \to \mathbf{R}$, $h : \Sigma \to \mathbf{R}$, find two real functions $u : Q \to \mathbf{R}$, $\gamma : (0, T) \to \mathbf{R}$, such that:

$$u_t - \Delta\beta(u) = 0 \qquad \text{in } Q \tag{2.1}$$

$$u(x, 0) = u_0(x) \qquad x \in \Omega \tag{2.2}$$

$$\beta(u(x,t)) = h(x,t) + \gamma(t) \qquad (x,t) \in \Sigma \tag{2.3}$$

$$\int_\Omega u(x,t)\,\mathrm{d}x = \mathrm{E}(t) \qquad 0 < t < \mathrm{T}. \tag{2.4}$$

From this formulation we pass to a new one. Integrating (2.1) over $\Omega \times (0,t)$ we obtain $\mathrm{E}(t) - \int_\Omega u_0\,\mathrm{d}x - \int_0^t \int_\Gamma \frac{\partial\beta(u)}{\partial n}\,\mathrm{d}\sigma = 0$ for $0 < t < \mathrm{T}$, and from here, deriving with respect to t, we have:

$$\int_\Gamma \frac{\partial\beta(u)}{\partial n}\,\mathrm{d}\sigma = \mathrm{E}'(t) \qquad \text{for} \quad 0 < t < \mathrm{T} \tag{2.5}$$

where n is the normal to Σ pointed into the outside of Q. We introduce the function Φ as follows:

$$-\Delta\Phi(t) = 0 \quad \text{in } \Omega, \qquad \Phi(t) = h(x,t) \quad \text{on } \Gamma. \tag{2.6}$$

Note that $\int_\Gamma \frac{\partial\Phi}{\partial n}\,\mathrm{d}\sigma = \int_\Omega \Delta\Phi\,\mathrm{d}x = 0$ for $t \in (0,\mathrm{T})$. Now we consider also the function l such that:

$$-\Delta l(t) = \mathrm{F}(t) \quad \text{in } \Omega, \qquad l(t) = 0 \quad \text{on } \Gamma \tag{2.7}$$

where $\mathrm{F}(t) = \mathrm{E}'(t)/\mu(t)$. Notice that $\int_\Gamma \frac{\partial l}{\partial n}(t)\,\mathrm{d}\sigma = \int_\Omega \mathrm{F}(t)\,\mathrm{d}\sigma = \mathrm{E}'(t)$ for $t \in (0,\mathrm{T})$.

Now we give an existence and uniqueness result for (Pb. 3), precising also the spaces of the data and the unknowns. We set $\mathrm{V} = \{v \in \mathrm{H}^1(\Omega),\ v_{|\Gamma} = \text{const.}\} \equiv \mathrm{H}^1_0(\Omega) \oplus \mathbf{R}$ and endow V with the norm of $\mathrm{H}^1(\Omega)$. If $\mathrm{H} \equiv \mathrm{L}^2(\Omega)$ and we identify H with its dual, we have $\mathrm{V} \subset \mathrm{H} \subset \mathrm{V}'$ with dense inclusions. We denote by $\mathrm{BV}(0,\mathrm{T};\mathrm{X})$ the space of functions with bounded variation from $(0,\mathrm{T})$ to X.

THEOREM 2. *Assume that* $\mathrm{F} \in \mathrm{L}^2(\mathrm{Q})$ *and* $\Phi, l \in \mathrm{BV}(0,\mathrm{T};\mathrm{H}^1(\Omega))$ (*these assumptions are certainly true if* $h \in \mathrm{BV}(0,\mathrm{T};\mathrm{H}^{\frac{1}{2}}(\Omega))$, $\mathrm{E}' \in \mathrm{BV}(0,\mathrm{T})$). *Assume* $u_0 \in \mathrm{L}^2(\Omega)$ *with* $j(u_0) \in \mathrm{L}^1(\Omega)$. *Then there exists a unique generalized solution* u *of* (Pb. 3), *i.e. such that* u, $\nabla(\beta(u)) \in \mathrm{L}^1(\mathrm{Q})$ *and such that for every* $w \in \mathrm{W}_0 \equiv \{w \in \mathrm{C}^1(\mathrm{Q}) : w(\mathrm{T}) = 0,\ w \in \mathrm{V}$ *for every* $t \in [0,\mathrm{T}]\}$:

$$-\iint_\mathrm{Q} u \frac{\partial w}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \iint_\mathrm{Q} \nabla(\beta(u) - \Phi - l)\,\nabla w\,\mathrm{d}x\,\mathrm{d}t = \iint_\mathrm{Q} \mathrm{F}w\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u_0\,w(0)\,\mathrm{d}x. \tag{2.8}$$

Moreover $u \in L^\infty(0, T; L^2(\Omega))$ *and* $(\beta(u) - \Phi - l) \in L^2(0, T; V)$.

Proof. In (2.8) the second term corresponds to a non-coercive operator in V. We can make it coercive adding the term $\iint_\Omega \beta(u)\, w \,dx\, dt$; since this is a Lipschitz-perturbation, we can repeat the proof of Theorem (2.1) and Proposition (5.2) of [6].

Hence we have the existence and the uniqueness of a weak solution $\theta = \beta(u) \in L^2(0, T; H^1(\Omega))$. Now we can introduce the freezing index as in (1.5). Set $\bar{\gamma}(t) = \int_0^t \gamma(\tau)\, d\tau$, $\bar{h}(x, t) = \int_0^t h(x, \tau)\, d\tau$, $L(t) = E(t) - \int_\Omega u_0(x)\, dx$. We find easily that w and γ satisfy the following equations ($\tilde{\chi} \in H(\theta)$ a.e. in Q):

$$w_t - \Delta w + \lambda \tilde{\chi} = u_0 \tag{2.9}$$

$$w(x, 0) = 0, \quad w_{|\Sigma} = \bar{h} + \bar{\gamma}, \quad \int_\Gamma \frac{\partial w}{\partial n}\, d\sigma = L(t) \quad \text{for } 0 < t < T. \tag{2.10}$$

The last condition is obtained integrating (2.9) in x over Ω. The problem of finding w which satisfies (2.9), (2.10) can be written as a variational inequality; for the details we still refer to [13]. Set $\mathscr{V} = L^2(0, T; H^1(\Omega))$, $\mathscr{K} = \{v \in \mathscr{V} : v_{|\Sigma} = h(x, t) + c(t),\ c \in L^2(0, T)\}$. $\mathscr{K}$ is a closed convex set in $\mathscr{V}$. The problem is the following:

(Pb. 4) given E, u_0, h as in Theorem 2, find $w \in \mathscr{V}$, $w' = \dfrac{\partial w}{\partial t} \in \mathscr{K}$ such that $w(x, 0) = 0$ and for every $v \in \mathscr{K}$:

$$\begin{aligned} &\iint_Q w'(v - w')\, dx\, dt + \iint_Q \nabla w \cdot \nabla (v - w')\, dx\, dt + \lambda \iint_Q v^+\, dx\, dt - \\ &- \lambda \iint_Q (w')^+\, dx\, dt \geq \iint_Q u_0 (v - w')\, dx\, dt + \int_0^T L(t)\, (v - w')_{|\Sigma}\, dt. \end{aligned} \tag{2.11}$$

THEOREM 3. *There exists one and only one solution of* (Pb. 4).

Proof. The existence part is ensured by Theorem 2. Uniqueness: notice that $w \in AC(0, T; H^1(\Omega))$. Suppose that there are two solutions w_1 and w_2, and let $w = w_1 - w_2$. We write (2.11) firstly for w_1 choosing $v = w_2'$, secondly for w_2 choosing $v = w_1'$. Then, adding the two inequalities, we get

$$\int_0^T\int_\Omega |w'(t)|^2\,dx\,dt + \int_0^T\int_\Omega \nabla w(t)\,\nabla w'(t)\,dx\,dt \le 0\,,\quad \text{hence}\quad \int_0^T\int_\Omega |w'(t)|^2\,dx\,dt +$$

$$+\frac{1}{2}\int_\Omega |\nabla w(\mathrm{T})|^2\,dx \le 0\,,$$ whence $w'(t)=0$ a.e. in $[0\,,\mathrm{T}]$, and this condition assures, with $w(0)=0$, that $w=0$.

Therefore, if u is the solution of (Pb. 3), the corresponding w is a solution of (Pb. 4): the two problems are equivalent because (Pb. 4) has a unique solution.

Remark 3. If we consider the one-phase problem, in the enthalpy formulation the only formal difference is that now we have $u_0 \ge 0$, $\mathrm{E}(t) \ge 0$, $h(x\,,t)+\gamma(t) \ge 0$, and β is defined by $\beta(x)=x-\lambda$ if $x>\lambda$, $\beta(x)=0$ if $x \le \lambda$. If we study the problem by the freezing index, we still get a variational inequality, which is simpler than (2.11). In fact in this case we have $\{(x\,,t)\in \mathrm{Q} : \theta(x\,,t)>0\}=\{x\,,t)\in \mathrm{Q} : w(x\,,t)>0\}$; then it is possible to multiply (2.11) by $(v-w)$ instead of $(v-w')$.

Remark 4. In the case of the one-phase one-dimensional Stefan problem it is sufficient to give the condition (2.4) without any Dirichlet condition, since there is already a condition on the boundary $\{x=c\}$ as $[\beta(u)]_x=0$. Taking w as unknown, we still obtain a variational inequality.

REFERENCES

[1] C. BAIOCCHI (1972) – *Su un problema a frontiera libera connesso a questioni di idraulica*, « Ann. Mat. Pura Appl. », (4) *92*, 107-127.

[2] J.R. CANNON (1963) – *The solution of the heat equation subject to the specification of energy*, « Quart. Appl. Math. », *21*, 155-160.

[3] J.R. CANNON and J. VAN DER HOEK (1981) – *The existence of and a continuous dependence result for the solution of the heat equation subject to the specification of energy*, « Boll. Un. Mat. It. », (4) *1*, 253-282.

[4] J.R. CANNON and J. VAN DER HOEK (1982) – *The one-phase Stefan problem subject to the specification of energy*, « J. Math. Anal. and Appl. », *86*, 281-291.

[5] J.R. CANNON and J. VAN DER HOEK (1982) – *The classical solution of the one-dimensional two-phase Stefan problem with energy specification*, « Ann. Mat. Pur. Appl. », (4) *130*, 385-398.

[6] A. DAMLAMIAN (1977) – *Some results on the multi-phase Stefan problem*, « Comm. Part. Diff. Equat. », *2* (10), 1017-1044.

[7] K.L. DECKERT and C.G. MAPLE (1963) – *Solution for diffusion with integral type boundary conditions*, « Proc. Iowa Acad. Sci. », *70*, 354-361.

[8] G. DUVAUT (1973) – *Résolution d'un problème de Stéfan*, « C.R. Acad. Sci. Paris » (A), *276*, 1461-1463.

[9] N.I. IONKIN (1977) – *Solution of a boundary value problem in heat conduction with a non-classical boundary condition*, « Differencial'nye Uravneija » *13*, 294-304 (English translation: « Differential Equations », *13* (2), 294-304).

[10] L.I. Kamynin (1964) – *A boundary value problem in the theory of heat conduction with a non classical boundary condition*, « Ž. Vyčisl. Mat. i Mat. Fiz. », *4*, 1006-1024 (English translation: « USSR Comp. Math. and Math. Phys. », *4* (6), 33-59).

[11] J.L. Lions and E. Magenes (1972) – *Non homogeneous boundary value problems and applications* vol. I, II, Springer, Berlin.

[12] E. Magenes (1976) – *Topics in parabolic equations: some typical free boundary problems*, Proc. N.A.T.O. Advanced Study for Evolution Partial Differential Equations, Liége, september 1976.

[13] E. Magenes (1981) – *Problemi di Stefan bifase in più variabili spaziali*, V S.A.F.A., Catania - « Le Matematiche », XXXVI, 65-108.