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Fisica dell'atmosfera. — *Some properties of forced, dissipative large-scale circulations in a barotropic, non-divergent rotating atmosphere.* Nota di LAURA GARDINI (*), RENZO LUPINI(**) e CARLO PELLACANI (***), presentata (****) dal Corrisp. E. BOSCHI.

RIASSUNTO. — Viene studiata la stabilità dell'atmosfera in un pianeta ruotante, forzata da un agente esterno ed in presenza di dissipazione. Lo studio vien condotto nelle ipotesi barotropiche e riguarda, per l'effetto delle approssimazioni adottate, solamente quei fenomeni caratterizzati da grandi scale spaziali. In particolare viene studiata la stabilità dei flussi zonali che caratterizzano la circolazione dei maggiori pianeti del sistema solare; ne vengono determinate, servendosi della approssimazione di Galerkin, le condizioni di stabilità asintotica globale.

1. We shall discuss in this note some properties of the large-scale circulations in a model of atmospheric dynamics which describes the action oƒ three main physical mechanisms: forcing due to an external source of energy (solar heating), dissipation due to decay to smaller scales of motion and non-linear exchange of energy among spectral components of the field of flow. The mathematical model is derived from the general dynamic equations for a rotating fluid by adding reasonable parametrizations of physical processes occurring at smaller scales of motion (convection, shear-instability, turbulent transfer, etc.) and by neglecting any phenomenon related to the vertical stratification of the atmosphere (non-divergent, two-dimensional barotropic flow is assumed) as well as the inhomogeneities of the planet's geometry (perfect sphericity is assumed). The resulting partial differential equation for the vorticity field is then further transformed, via a Galerkin-type approximation, into a finite set of ordinary differential equations which describe the time evolution of the amplitudes of spectral components of the field of motion, representative of relevant prototypes of the global barotropic circulation, that is steady, axisymmetric (or zonal) flow and periodic Rossby-Haurwitz waves. Thus, the climatic problem, concerning the long-time behaviour of the planetary circulation, is formulated in terms of recurrence and stability properties of the solution's curves in the phase-space of the model, as a function of a finite number of control parameters (forcing intensity, dissipation time-scales and angular speed of absolute rotation).

In particular we study the global and local asymptotic stability (in the sense of Liapounoff) of particular solutions representing purely zonal circulations

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for time-dependent, axisymmetric forcing field. In the autonomous case (constant forcing) the model defines a dynamical system with a compact, global attractor set whose critical elements (fixed points and periodic orbits) are the counterpart of steady, zonal circulations and periodic wave-oscillations, respectively. The basins of attraction of the fixed points are investigated, as a function of the control parameters; in particular we analyse the occurrence of bifurcations of such fixed points into periodic orbits in the neighbourhood of their stability boundary.

2. The Navier-Stokes equations for two-dimensional flows in a layer of homogeneous, incompressible fluid on a rotating, spherical planet can be written (Pedlosky, 1979)

$$\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}(v \cos \theta) = 0 \tag{2.1}$$

$$\rho\left(\frac{du}{dt} - \frac{uv}{a} \tan \theta - 2\,\omega \sin \theta\, v + \frac{\partial P}{\partial x}\right) = g_x + f_x \tag{2.2}$$

$$\rho\left(\frac{dv}{dt} + \frac{u^2}{a} \tan \theta + 2\,\omega \sin \theta\, u + \frac{\partial P}{\partial y}\right) = g_y + f_y \tag{2.3}$$

where a is the planet's radius, φ and θ are the longitude and the latitude, respectively, $dx = a \cos \theta \, d\varphi$ and $dy = a \, d\theta$ are the differentials of the corresponding curvilinear coordinates, ω is the angular speed of rotation, P is the pressure, (u, v) is the velocity field, ρ is the density, (g_x, g_y) is the field of external forces and (f_x, f_y) is the field of internal (frictional) forces. Moreover,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{1}{a \cos \theta} \frac{\partial}{\partial \varphi} + \frac{v}{a} \frac{\partial}{\partial \theta}$$

is the 'convective' time-derivative.

The continuity equation (2.1) can be solved to give

$$(u, v) = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}\right)$$

where the stream function $\psi = \psi(\varphi, \mu, t)$, $\mu = \sin \theta$, is any smooth function such that $\psi(\varphi, \pm 1, t) = 0$. Noting that the 'geometric' terms $\frac{uv}{a} \tan \theta$ and $\frac{u^2}{a} \tan \theta$ are bounded as θ tends to $\pm(\pi/2)$, the limiting case of circulations with small relative angular momentum, that is $(|u| + |v|) \ll a\omega$, is described by the same set of equations (2.2), (2.3) without geometric terms. For such motions we can reduce these equations, by cross differentiation and summation, to a single scalar equation, whose non-dimensional form (we take

ω^{-1} and a as the scales for time and space, respectively) reads

$$\frac{\partial}{\partial t}\zeta + J(\zeta, \psi) + 2\frac{\partial \zeta}{\partial \theta} = G + F \tag{2.4}$$

where $\zeta = \nabla^2 \psi$ is the vorticity and $J(\zeta, \psi)$ denotes the Jacobian of ζ and ψ. G and F represent the vorticity generation and dissipation fields, respectively.

Introducing the sequence of spherical harmonic functions $Y_n^l(\varphi, \mu)$ as a complete orthonormal set, we can write the developements

$$\psi = -\sum_{n=1}^{\infty} \sum_{l \leq |n|} c_n \psi_n^l(t) Y_n^l \tag{2.5}$$

$$\zeta = \sum_{n=1}^{\infty} \sum_{l \leq |n|} \zeta_n^l(t) Y_n^l \tag{2.6}$$

where $C_n = 1/n(n+1)$. Here use has been made of the well-known spectral property $\nabla^2 Y_n^l + n(n+1) Y_n^l = 0$.

We can call l and n the zonal and the meridional index, respectively, of a spectral component of flow $\psi_n^l Y_n^l$, as the number of nodal meridians of Y_n^l is $2|l|$ and the number of nodal parallels is $n - |l|$ (see Abramovitz and Stegun, 1972).

Introducing the developements (2.5), (2.6) into the vorticity equation (2.4) and equating the Fourier coefficients of both members, the following sequence of ordinary differential equations is obtained (Platzman, 1960; Dutton, 1982)

$$\dot{\zeta}_\gamma = \frac{1}{2}\sum_{\beta,\alpha} I_{\gamma\beta\alpha}\zeta_\beta \zeta_\alpha - i\omega_\gamma \zeta_\gamma + G_\gamma + F_\gamma \tag{2.7}$$

$$\gamma = (n_\gamma, l_\gamma)\ ; \quad n_\gamma = 1, 2, \ldots ; \quad |l_\gamma| \leq n_\gamma$$

where we use the notation $\zeta_\gamma = \zeta_{n_\gamma}^{l_\gamma}$ and $\omega = l_\gamma \omega(1 - 2c_\gamma)$, while the explicit representation of the 'interaction' coefficients $I_{\gamma\beta\alpha}$ is reported in the appendix. In the following we shall use for the dissipation terms the simple representation $F_\gamma = -\nu_\gamma \zeta_\gamma$ where the 'friction' coefficients ν_γ (positive numbers) measure the inverse of the decay-times of the various spectral components in the absence of forcing. Sometimes, in order to simplify the mathematical treatment, we shall also assume a unique dissipation time-scale, that is $\nu_\gamma = \nu$ for any γ.

3. Clearly, the infinite sequence of equations (2.7), which is equivalent to the original partial differential equation (2.4), describes motions on all space-time scales, so that we have to 'truncate' it in order to isolate the dynamics of the planetary-scale components of the field of flow. The guiding principles for such a truncation are mathematical simplicity and, so far as possible, qualitative adherence to observed properties of the planetary circulations. First. we identify within the spectrum those flow-components which represent the basic prototypes of the averaged atmospheric circulation. We note that

$\sum_{n=1}^{\infty} \psi_n Y_n$ represents the axisymmetric part of the field of flow (no longitudinal structure); for an axisymmetric forcing field, such as the one induced by solar heating, we expect that kinetic energy is primarily fed into purely zonal circulations, so that we have to include in our model a representative set of large-scale (small n) axisymmetric components. Next, by non-linear interaction, there will be a transfer of energy from zonal flow to wave-components, $\psi_{n_\gamma}^{l_\gamma} Y_{n_\gamma}^{l_\gamma}$, $l_\gamma \neq 0$, followed by a complicated 'cascade' of energy through the whole wave-spectrum. We shall restrict our attention to the study of the first two mechanisms only, so that we shall discard in the basic set of equation (2.7) all the non-linear interactions among wave components, whereas we shall retain all possible wave-zonal flow couplings. By the selection rules obeyed by the interaction coefficients such truncation procedure is equivalent to reducing the spectral distribution of the indices to the two sequences $\{(n, 0); n = 1, 2, \ldots . s\}$, which identifies a section of the zonal-flow field, and $\{(n, l); n = 1, 2, \ldots, w\}$ which identifies the wave-components of given longitudinal structure interacting with the zonal field of motion.

The resulting truncated set of $s + w = m$ ordinary differential equations is

$$\dot{\zeta}_n = 2 \sum_{\gamma > \beta} (I_{\gamma\beta n} - I_{\beta\gamma n}) \mathscr{I}_m (\zeta_\beta \zeta_\gamma^*) + G_n - \nu_n \zeta_n \qquad n = 1, 2, \ldots s \tag{3.1}$$

$$\dot{\zeta}_\beta = i \left(\sum_n I_{\beta\beta n} \zeta_n - \omega_\beta \right) \zeta_\beta + i \sum_{\gamma \neq \beta} \sum_n I_{\gamma\beta n} \zeta_n \zeta_\gamma + G_\beta - \nu_\beta \zeta_\beta \tag{3.2}$$

$$n_\beta = 1, 2, \ldots, w$$

where $\mathscr{I}_m$ denotes the imaginary part, i is the imaginary unit, $\boldsymbol{n} = (n, 0)$ and $\omega_\beta = \omega l (1 - 2 C_{n_\beta})$. Note that the vector index for wave-components $\gamma = (n_\gamma, l)$ can be identified with the scalar index n_γ, by constancy of l.

A more compact formulation of the above set of equations can be obtained by introducing the following vector notations

$$\begin{aligned} (\mathbf{I}_{\gamma\beta})_n &= I_{\gamma\beta n} \\ (\boldsymbol{x})_\beta &= \zeta_\beta \\ (\zeta)_n &= \zeta_n \\ (\mathbf{G})_n &= G_n \\ (\boldsymbol{h})_\beta &= G_\beta . \end{aligned} \tag{3.3}$$

The set of equations (3.1), (3.2) can be re-written as

$$\dot{\zeta} = 2 \sum_{\gamma > \beta} (\mathbf{I}_{\gamma\beta} - \mathbf{I}_{\beta\gamma}) \mathscr{I}_m (\zeta_\beta \zeta_\gamma^*) + \mathbf{G} - N\zeta \tag{3.4}$$

$$\dot{\boldsymbol{x}} = A(\zeta) \boldsymbol{x} + \boldsymbol{h} \tag{3.5}$$

where $N = \mathrm{diag}\{\nu_1, \nu_2, \dots, \nu_s\}$ and $A = i(M - \Omega) - N_1$ is the $w \times w$ matrix defined by

$$
\begin{aligned}
[M]_{\gamma\beta} &= \mathbf{I}_{\gamma\beta} \cdot \zeta \\
[\Omega]_{\gamma\beta} &= \omega_\beta\, \delta_{\gamma\beta} \\
[N_1]_{\gamma\beta} &= \nu_\beta\, \delta_{\gamma\beta}\,.
\end{aligned}
\tag{3.6}
$$

4. It is clear that the conditions ensuring local existence and uniqueness of the solutions of the set (3.4). (3.5) are fulfilled throughout the whole space $\mathbf{R}_m$, when $\mathbf{G}$ and $\boldsymbol{h}$ are continuously differentiable functions of time. We show in this section that, in addition, for bounded forcing vectors $\mathbf{G}$ and $\boldsymbol{h}$, the solution curves are global in the future (see Stepanov and Nemytskii, 1960), as they enter, after a finite time, a positively-invariant, closed neighbourhood of the origin.

Let us consider the evolution equation for the two positive, quadratic forms $V = \frac{1}{2} \| \zeta \|^2 + \| \boldsymbol{x} \|^2$ and $W = \| \boldsymbol{x} \|^2$. Taking the scalar product of equations (3.4) and (3.5) by ζ and $\boldsymbol{x}$ respectively, and using the symmetry property of the interaction coefficients (see appendix) we find

$$\dot{V} = -2\, N_0\, \boldsymbol{v} \cdot \boldsymbol{v}^* + 2\, \mathcal{R}e\, \boldsymbol{\varphi} \cdot \boldsymbol{v}^* \tag{4.1}$$

$$\dot{W} = (A + A^*)\, \boldsymbol{x} \cdot \boldsymbol{x}^* + 2\, \mathcal{R}e\, \boldsymbol{h} \cdot \mathrm{x}^* \tag{4.2}$$

where $N_0 = N + N_1$, $\boldsymbol{v} = \left(\frac{1}{\sqrt{2}}\, \zeta, \boldsymbol{x} \right)$ and $\boldsymbol{\varphi} = (\mathbf{G}, \boldsymbol{h})$.

Moreover, $\mathcal{R}e$ means real part, A^* is the adjoint of A and the dot between vectors denotes scalar product.

By equation (4.1) the following inequality is deduced

$$\frac{1}{2}\, \dot{V} \leq -\nu_*\, V + \| \boldsymbol{\varphi} \|\, V^{1/2} \tag{4.3}$$

where $\nu_* = \min\limits_k N_{0kk}$. Equation (4.3) can be integrated to give

$$V^{1/2} \leq V^{1/2}(0) \exp(-\nu_* t) + \int_0^t \| \boldsymbol{\varphi} \| \exp(\nu_* (t' - t))\, dt'\,. \tag{4.4}$$

For a bounded forcing function, that is $\sup\limits_{t \geq 0} \| \boldsymbol{\varphi} \| \leq \mu$, equation (4.4) gives

$$V^{1/2} \leq \left(V^{1/2}(0) - \frac{\mu}{\nu_*} \right) \exp(-\nu_* t) + \frac{\mu}{\nu_*}\,. \tag{4.5}$$

Let C be the convex, compact subset of $\mathbf{R}_m$ defined by $V^{1/2} \leq \frac{\mu}{\nu_*}$.

By equation (4.5) solution curves starting within C remain there indefinitely, while, trajectories starting outside C enter an ε-neighbourhood of C, V_ε, of equation $V^{1/2} < \frac{\mu}{\nu_*} + \varepsilon$ in a finite time, T_ε, given by

$$T_\varepsilon = -\frac{1}{\nu_*} \ln \frac{\varepsilon}{V^{1/2}_{(0)} - \frac{\mu}{\nu_*}} .$$

We can then state the following

PROPOSITION 4.1. *For a bounded forcing function there is an invariant, closed neighbourhood of the origin, say* C, *which is globally attracting in the future for the solutions of the set* (3.1), (3.2).

An important consequence of this proposition is that, for constant forcing, the long-term (climatic) behaviour of the atmospheric circulation, modelled by the system of equations (3.1), (3.2), is described by the recurrence properties of the corresponding dynamical system (Stepanov and Nemytskii, 1960) restricted to the compact subset C of the phase space.

From now on we shall restrict our analysis to the case of purely axisymmetric forcing, that is, we shall take $\boldsymbol{h} = 0$ in the basic set of equations (3.1) (3.2). Clearly, the relevant solution is now $\zeta(t) = N^{-1} \mathbf{G}(t)$, $\boldsymbol{x} = 0$, which represents a state of forced zonal flow (FZF). We shall study the conditions for which this FZF-solution is globally, asymptotically stable, that is $\lim_{t \to +\infty} W = 0$, for any initial condition.

Let $\sigma(t)$ denote the maximum of the (time-varying) spectrum of the matrix $A + A^*$. By equation (4.2) we can write ($\boldsymbol{h} = 0$)

$$W^{1/2} \leq W^{1/2}(0) \exp S(t) \tag{4.6}$$

where $S(t) = \int_0^t \sigma(t')\, dt'$. It is clear that $S(t) \leq -\alpha < 0$ is a sufficient condition for global, asymptotic stability of FZF.

Now, by Gerschgorin's theorem (see Wilkinson, 1965) any eigenvalue of $A + A^*$ must satisfy the inequality

$$|\lambda - 2\nu_\beta| \leq \sum_\gamma |(\mathbf{I}_{\beta\gamma} - \mathbf{I}_{\gamma\beta}) \cdot \zeta| ; \qquad \forall \beta .$$

It follows that

$$\max_{\beta} \sum_{\gamma} \| (\mathbf{I}_{\beta\gamma} - \mathbf{I}_{\gamma\beta}) \| \quad \| \zeta \| < 2\, \nu_\beta\,, \qquad \forall t \geq 0 \tag{4.7}$$

is sufficient for global, asymptotic stability of FZF. On the other hand, by equation (4.5) and Proposition 4.1, for any solution starting within C, we have $\| \zeta \| \leq \sqrt{2}\ V^{1/2} \leq \frac{\mu}{\nu_*}$, so that

$$\mu < \frac{2^{3/2}\, \nu_*^2}{\max_{\beta} \sum_{\gamma} \| \mathbf{I}_{\beta\gamma} - \mathbf{I}_{\gamma\beta} \|} \tag{4.8}$$

implies (4.7). We can then state the following.

PROPOSITION 4.2. *In the case of weak (in the sense of equation* (4.8)) *axisymmetric* ($\boldsymbol{h} = 0$) *forcing, the corresponding* FZF*-solution,* $\zeta = \mathrm{N}^{-1}\,\mathbf{G}$ *and* $\boldsymbol{x} = 0$, *of the system* (3.1), (3.2) *is globally, asymptotically stable.*

5. We note that the condition for global, asymptotic stability of FZF given in the previous section does not involve the angular speed of rotation, which has been eliminated, during the demonstration, by the rough estimate given in equation (4.6).

In order to establish the role of absolute rotation on the stabilty of forced axisymmetric circulations, we have then to resort to a weaker stability condition, such as the local, asymptotic stability condition. In this section we shall assume that the axisymmetric forcing field is steady, so that the set of equations (3.1), (3.2) is autonomous and the FZF-solution represents a fixed point for the corresponding dynamical system. Thus, the local, asymptotic stability condition can be written

$$\mathcal{R}e\, \lambda_A < 0 \tag{5.1}$$

where λ_A denotes the spectrum of A. To simplify our analysis we shall also assume $N_1 = \nu E$, where E is the identity matrix $w \times w$. Equation (5.1) defines an open domain, say S, in $\mathbf{R}_s$, whose boundary (the ' marginal ' stability boundary) we shall call Σ. The open set $U = S' - \Sigma$ contains all the vectors $\zeta = N^{-1}\,\mathbf{G}$ such that the corresponding FZF-solution is unstable. By Proposition 4.2, for any vector $\boldsymbol{e}$ belonging to $\mathbf{R}_s$ there exist $\delta > 0$ such that the segment $\{\xi\, \boldsymbol{e};\ |\ \xi\ | < \delta\}$ belongs to S. We can decompose $\boldsymbol{e}$ into the sum of two vectors, $\boldsymbol{e}^+$ and $\boldsymbol{e}^-$ say, such that $M^+ = M(\boldsymbol{e}^+)$ is a symmetric matrix, while $M^- = M(\boldsymbol{e}^-)$ is antisymmetric (see equations 3.5). Thus, the matrix $A(\xi\, \boldsymbol{e})$ can be written in the form

$$i\xi M^- + i\,(\xi M^+ + \Omega) - \nu E\,. \tag{5.2}$$

Clearly, if $M^- = 0$ the whole line through $\boldsymbol{e}$ belongs to S, whereas if $M^- \neq 0$ it crosses Σ in at least one point. In order to examine the effect of absolute rotation on the stability of FZF-solutions, we shall determine the perturbations on Σ induced by slight variations of the matrix Ω. In particular we assume that the spectrum of $i\,\xi^* M^-$ is critical ($\zeta^* \boldsymbol{e} \in \Sigma$), that is

$$\xi^* \lambda_{iM^-} = \{\lambda_1, \lambda_2, \ldots, \lambda_w\} \subset \mathbf{R} \tag{5.3}$$

with $\lambda_1 = \nu > \lambda_2 > \ldots > \lambda_w$, and we look for the spectrum of A when $\xi^* M^T - \Omega = \varepsilon D$, D a symmetric matrix and $|\varepsilon| \ll \nu$.

We can apply ordinary perturbation theory (Wilkinson, 1965) to find the coefficients of the expansion

$$\lambda = \nu + \varepsilon\,\mu_1 + \varepsilon^2\,\mu_2 + 0\,(\varepsilon^3)$$

representing the perturbed 'critical' eigenvalue λ_1. After direct computation of the perturbation formulae, we find

$$\mu_1 = 0 \tag{5.4}$$

$$\mu_2 = \sum_{k=2}^{w} \frac{|\,D\boldsymbol{x}_1 \cdot \boldsymbol{x}_k^*\,|^2}{\lambda_k - \nu} < 0 \tag{5.5}$$

where $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_w\}$ is the orthonormal set of eigenvectors of iM^- corresponding to the spectrum of equation (5.1). Equations (5.4), (5.5) imply that the marginally stable FZF-fixed point $\xi^* \boldsymbol{e}$ becomes stable under slight, symmetric perturbations of the basic matrix A; however, it is also clear from equation (5.2) that the stabilizing effect of absolute rotation is strongly coupled with the structure itself of the forced zonal flow, as this structure influences the form of $M^+ + \Omega$ and, consequently the value of the right side of equation (5.2). To clarify this point let us consider the case of a single dyadic wave-field, in which case the matrix A is 2×2 and the stability domain S can be expressed analytically by the equation

$$(\boldsymbol{b} \cdot \zeta + \Delta\omega)^2 + (\mathbf{I}_{\gamma\beta} \cdot \zeta)\,(\mathbf{I}_{\beta\gamma} \cdot \zeta) + 4\,\nu^2 > 0 \tag{5.6}$$

where $\boldsymbol{b} = \mathbf{I}_{\beta\beta} - \mathbf{I}_{\gamma\gamma}$ and $\Delta\omega = \omega_\gamma - \omega_\beta$. Clearly the marginal stability boundary Σ is an elliptic hyperboloid, in this case. The form of the first term on the left side of equation (5.6) shows that absolute rotation (proportional to $\Delta\omega$) is stabilizing for only those FZF-structures such that $0 \leq \Delta\omega\,\boldsymbol{b} \cdot \zeta$. In particular, absolute rotation stabilizes anti-symmetric (with respect to the planet's Equator) fields of zonal flow, as, in this case $\boldsymbol{b} \cdot \zeta = 0$ (see selection rules).

In this case of dyadic wave fields, the effect of solid rotation on the stability boundary admits a global geometric description; in fact, denoting by Σ_ω the marginal stability boundary corresponding to ω, we easily find, by equation (5.6), $\Sigma_\omega = \boldsymbol{c} + \Sigma_0$, where the translation vector $\boldsymbol{c}$ is the solution of the linear set of equations $\boldsymbol{b} \cdot \boldsymbol{c} + \Delta\omega = 0$, $\mathbf{I}_{\gamma\beta} \cdot \boldsymbol{c} = \mathbf{I}_{\beta\gamma} \cdot \boldsymbol{c} = 0$.

6. When the constant, forcing vector $\mathbf{G}$ is changed so that the corresponding FZF-fixed point ($\zeta = \mathrm{N}^{-1}\,\mathbf{G}$, $\boldsymbol{x} = 0$) crosses the marginal stability boundary Σ and becomes unstable, a Hopf-bifurcation is expected to occur, in general (Marsden and McCracken, 1976). For our system of equations (3.4), (3.5) a particular family of periodic orbits can be found in a neighbourhood of Σ, and represents an important new type of unsteady, forced regime of large-scale flow.

Let ζ_0 belong to Σ; then there exists a real number σ_0 and a vector $\boldsymbol{x}_0$ such that

$$\mathrm{A}(\zeta_0)\,\boldsymbol{x}_0 = i\sigma_0\,\boldsymbol{x}_0\,. \tag{6.1}$$

Let us consider the family of periodic (circular) orbits of equations $\zeta = \zeta_0$ and $\boldsymbol{x} = \alpha\boldsymbol{x}_0 \exp(i\sigma_0 t)$, where α is any complex number. Clearly, these orbits satisfy equation (3.5).

The also satisfy equation (3.4) if

$$2\,|\,\alpha\,|^2 \sum_{\gamma>\beta} (\mathbf{I}_{\gamma\beta} - \mathbf{I}_{\beta\gamma})\,\mathscr{I}_m(\zeta_\beta^0\,\zeta_\gamma^{0*}) = \mathrm{N}\zeta_0 + \mathbf{G} \tag{6.2}$$

where ζ_β^0 denotes $(\boldsymbol{x}_0)_\beta$. In order to solve equation (6.2) with respect to α we multiply equation (6.1) by $\boldsymbol{x}_0$ to obtain

$$-\sum_{\gamma>\beta} (\mathbf{I}_{\gamma\beta} - \mathbf{I}_{\beta\gamma}) \cdot \zeta_0\,\mathscr{I}_m(\zeta_\beta^0\,\zeta_\gamma^{0*}) = (\mathrm{N}_1\,\boldsymbol{x}_0 \cdot \boldsymbol{x}_0) > 0\,. \tag{6.3}$$

By equations (6.2), (6.3) the following condition for the existence of circular orbits is obtained

$$\zeta_0 \cdot (\mathrm{N}\zeta_0 - \mathbf{G}) < 0\,. \tag{6.4}$$

Equation (6.4), given ζ_0 belonging to Σ, defines a half space, say Γ_0, of bifurcation vectors $\mathrm{N}^{-1}\,\mathbf{G}$ bounded by an hyperplane, say P_0, passing through the critical vector $\mathrm{N}^{-1}\,\mathbf{G}_0 = \zeta_0$. As, typically, P_0 is transverse to Σ at ζ_0, then $\Gamma_0 \cap \mathrm{U}$ is a non-void, open set. By considering the reunion $\Gamma = \bigcup_{\zeta_0 \in \Sigma} (\Gamma_0 \cap \mathrm{U})$ we can formulate the following

PROPOSITION 6.1. *In the space of (constant) forcing vectors there is an open subset of* U *bounded by* Σ *to which there correspond unstable* FZF*-fixed points and periodic orbits of the form* $\zeta = cte.$, $\boldsymbol{x} = \alpha\boldsymbol{x}_0 \exp(i\sigma_0 t)$ *with* α *and* $\boldsymbol{x}$ *complex and constant,* σ_0 *real and constant.*

The local, asymptotic stability of such circular orbits can be proved in the simple case of a dyadic wave-field (Lupini and Pellacani, 1984). For the general case we have only numerical evidence of stability but no rigorous proof.

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Appendix

Letting $Y_\gamma(\varphi, \mu) = Y_{n_\gamma}^{l_\gamma}(\varphi, \mu) = P_{n_\gamma}^{l_\gamma}(\mu)\, e^{il_\gamma \varphi}$ (Abramovitz and Stegun, 1972), the interaction coefficients $I_{\gamma\beta\alpha}$ can be expressed in the following form (Platzman, 1960)

$$I_{\gamma\beta\alpha} = \begin{cases} 0\,; & l_\gamma \neq l_\alpha + l_\beta \\ (c_\beta - c_\gamma) \displaystyle\int_{-1}^{1} P_\gamma \left(l_\gamma P_\beta \frac{dP_\alpha}{d\mu} - l_\alpha P_\alpha \frac{dP_\beta}{d\mu} \right) d\mu\,. & \end{cases}$$

By well-known properties of the Legendre functions $P_n^l(\mu)$ the following selection rules hold: $I_{\gamma\beta\alpha} = 0$ unless

$$| n_\alpha - n_\beta | < n_\gamma < n_\alpha + n_\beta$$
$$n_\alpha + n_\beta + n_\gamma = \sigma\, dd$$
$$l_\alpha^2 + l_\beta^2 \neq 0$$
$$l_\gamma = l_\beta + l_\alpha\,.$$

Moreover, the following symmetry relation holds

$$I_{\gamma\beta\alpha} + I_{\beta\gamma\alpha} = 0$$

for $l_\alpha = 0$.