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# Adiabatic invariants for a collisional magnetoplasma in the presence of spatial symmetries

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**Fisica matematica.** — *Adiabatic invariants for a collisional magnetoplasma in the presence of spatial symmetries* (*). Nota di MASSIMO TESSAROTTO (**), presentata (***) dal Socio D. GRAFFI.

RIASSUNTO. — Si studiano, nella cosiddetta approssimazione di « piccolo raggio di Larmor », le proprietà di soluzioni perturbative dell'equazione di Fokker–Planck che descrive un magnetoplasma quiescente immerso in una configurazione di equilibrio idromagnetico spazialmente simmetrica.

Si formula il problema della determinazione degli invarianti adiabatici ammessi da tale sistema e si dimostra che, oltre agli usuali, il plasma ne ammette altri dovuti specificatamente alla simmetria.

## 1. INTRODUCTION

It is the purpose of the present Note to point out some basic properties of perturbative solutions of the Fokker-Planck equation obtained in the framework of the so-called « small Larmor radius ordering » (SLRO), previously described by various authors [1-3], and which is currently adopted for investigations on collisional transport problems in magnetoplasmas [4-7].

In particular we intend to analyze the existence of adiabatic invariants, which in analogy with the case of so-called « collisionless » plasmas [8], corresponds to appropriate choices of the initial condition for the one-particle distribution function.

We find that not all of the first-order adiabatic invariants occurring for collisionless systems are recovered in a collisional plasma. In particular, while the kinetic energy and the Euler potentials, defining the magnetic surfaces, result adiabatic invariants, under very strong requests on the choice of the « equilibrium » distribution, the magnetic moment (also called « perpendicular » adiabatic invariant) and the so-called parallel adiabatic invariant are no longer conserved.

The consequences of assuming a spatial symmetry for the electromagnetic fields and the one-particle distribution function are then investigated.

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Three new adiabatic invariants are pointed out which are related respectively to the canonical momentum of a given particle and conjugated to the cyclic coordinate, to the analogous canonical momentum of the plasma (i.e., obtained by taking the moment of the previous one in terms of the one-particle distribution function) and to the relative mean velocity of the various plasma species at equilibrium, as corresponds to a magnetoplasma in the presence of non-rigid plasma motions (1).

Since such dynamical variables are (at least) second-order invariants, we conclude that they vary on a time-scale comparable (or even larger) with the typical collisional time scales [4]. An important consequence of these results is that variational principles for the Fokker-Planck kinetic equation, relevant to describe such initial states should, in fact, be appropriately generalized [4, 10, 11] to such a broader class of equilibria, by taking into account such invariants. Such a programme shall be carried out in a forthcoming paper.

## 2. Adiabatic invariants in a collisional magnetoplasma

The concept of adiabatic invariant, as is well known in the literature [12, 13], arises naturally in the context of guiding centre or guiding center drift theories. Such a type of theories, which are strictly applicable only to collisionless magnetoplasmas, rely heavily on the necessity of determining either the test particle or the guiding center orbit [14].

On the contrary, in a collisional magnetoplasma since collisions may influence also the guiding centre orbits, such theories are generally not applicable. In fact it is well known, for example, that $\mu = v_\perp^2/2\,\mathrm{B}$ in the presence of collisions is not an adiabatic invariant (here $v_\perp$ denotes, as usual, $v_\perp = |\boldsymbol{v}_\perp|$, with $\boldsymbol{v}_\perp = \boldsymbol{v} - \boldsymbol{n}v_\parallel$ being $\boldsymbol{n} = \mathbf{B}/\mathrm{B}$).

In the sequel, in order to investigate the problem of the determination of adiabatic invariants in a collisional plasma, we shall, adopt, instead, a perturbative expansion based on a " small Larmor radius ordering " (SLRO), previously introduced by various authors [1 - 3] and following a method analogous to that developed originally by Hastie, Taylor and Haas to investigate a Vlasov (i.e., a collisionless) magnetoplasma [8].

We recall, for definition, the basic assumptions of SLRO, which can be defined in terms of characteristic time scales, resulting from the inverse of the following frequencies: $\Omega_s = e_s\,\mathrm{B}/m_s\,c$ (the Larmor frequency), $\omega_b = \left(\oint\limits_{\mathrm{C}} \mathrm{d}l/|v_\parallel|\right)^{-1}$ (the bounce or transit frequency, characterizing the unperturbed motion along a magnetic flux line), $\omega_{\mathrm{D}b,s} = \left\{\mathrm{L}\oint\limits_{\mathrm{C}} \mathrm{d}l\, v_{\mathrm{D},s}/|v_\parallel| / \oint\limits_{\mathrm{C}} \mathrm{d}l/|v_\parallel|^{-1}\right\}$ (the drift-bounce

(1) Such a type of equilibria has been investigated in detail in Ref. [9], henceforth called simply « I ».

frequency, characterizing the motion of the guiding center along a magnetic flux line), $\nu_{s,\mathrm{eff}}$ (an appropriate effective collision frequency), $\omega_{p,s} = (4\pi N_s e_s^2/m_s)^{\frac{1}{2}}$ (the plasma frequency). Thus we impose [1, 2]:

$$(1) \qquad \omega_b \sim \Omega_s \varepsilon_s \quad ; \quad \nu_{s,\mathrm{eff}} \sim \Omega_s \varepsilon_s \quad ; \quad \Omega_{\mathrm{D}b,s} \sim \Omega_s \varepsilon_s^2 \quad ; \quad \omega_{pe} \sim \Omega_e \varepsilon_e^0$$

being $\varepsilon_s = r_s/\mathrm{L}$, with $r_s = v_{th,s}/\Omega_s$ (where $v_{th,s}$ is the thermal velocity) the Larmor radius and L a characteristic scale length at equilibrium (see also I for its definition).

The Fokker-Planck kinetic equation and the Maxwell's equations for the electro-magnetic field are then solved by expanding all the physically relevant quantities in power series of $\varepsilon_s$, assuming that solutions can be found of the form, form example in the case of the one-particle distribution function $f_s(\boldsymbol{r}, \boldsymbol{v}, t)$:

$$(2) \qquad f_s(\boldsymbol{r}, \boldsymbol{r}, t) = f_{0,s}(\boldsymbol{r}, \boldsymbol{v}, t_0, t_1, \ldots) + \varepsilon_s f_{1,s}(\boldsymbol{r}, \boldsymbol{v}, t_0, t_1, \ldots) + \ldots$$

where the variables $t_0, t_1, \cdots$ are related to the time $t$ by the equations $\mathrm{d}t_i/\mathrm{d}t = \varepsilon_s^i \; (i = 0, 1, 2, \cdots)$. The Fokker-Planck equation thus delivers the hierarchy of perturbative equations:

$$(3) \qquad \mathrm{L}_{i,s}(f) = 0$$

where, in particular, results:

$$\mathrm{L}_{0,s} = \frac{\partial}{\partial t_0} + \boldsymbol{v} \wedge \boldsymbol{\Omega}_{0,s} \cdot \frac{\partial}{\partial \boldsymbol{v}}$$

$$(4) \qquad \mathrm{L}_{1,s}(f) = \frac{\partial}{\partial t_0} f_{1,s} + \frac{\partial}{\partial t_1} f_{0,s} + \boldsymbol{v} \cdot \nabla f_{0,s} + \frac{e_s}{m_s} \mathbf{E}_0 \cdot \frac{\partial}{\partial \boldsymbol{v}} f_{0,s} + \\ + \boldsymbol{v} \wedge \boldsymbol{\Omega}_{0,s} \cdot \frac{\partial}{\partial \boldsymbol{v}} f_{1,s} + \boldsymbol{v} \wedge \boldsymbol{\Omega}_{1,s} \cdot \frac{\partial}{\partial \boldsymbol{v}} f_{0,s} - \mathrm{C}_s(f_0 \mid f_0)$$

$$\mathrm{L}_{2,s}(f) = \frac{\partial}{\partial t_0} f_{2,s} + \frac{\partial}{\partial t_1} f_{1,s} + \frac{\partial}{\partial t_2} f_{0,s} + \boldsymbol{v} \,.\, \nabla f_{1,s} + \frac{e_s}{m_s} \mathbf{E}_0 \cdot \frac{\partial}{\partial \boldsymbol{v}} f_{1,s} + \\ + \frac{e_s}{m_s} \mathbf{E}_1 \cdot \frac{\partial}{\partial \boldsymbol{v}} f_{0,s} + \boldsymbol{v} \wedge \boldsymbol{\Omega}_{0,s} \cdot \frac{\partial}{\partial \boldsymbol{v}} f_{2,s} + \boldsymbol{v} \wedge \Omega_{1,s} \cdot \frac{\partial}{\partial \boldsymbol{v}} f_{1,s} + \\ + \boldsymbol{v} \wedge \Omega_{2,s} \cdot \frac{\partial}{\partial \boldsymbol{v}} f_{0,s} - \mathrm{C}_s(f_0 \mid f_1)$$

where $\boldsymbol{\Omega}_{i,s} = e_s \mathbf{B}_i/m_s c$ and $\mathbf{B}_i$ denotes the contribution of order $0(\varepsilon_s^i)$ to $\mathbf{B}$.

We look for particular solutions of Eqs. (3) yielding a so-called "minimal diffusion", i.e., we require that $f_s(\boldsymbol{r}, \boldsymbol{v}, t)$ be constant in time to an order as high as possible, in the sense that results $\left(\frac{\partial}{\partial t} f_s\right)_i = 0$ for $i = 1, 2, \cdots k$ with $k$ the maximum allowed by Eqs. (3) [2], being respectively $\left(\frac{\partial}{\partial t} f_s\right)_0 = \frac{\partial}{\partial t_0} f_{0,s}$, $\left(\frac{\partial}{\partial t} f_s\right)_1 = \frac{\partial}{\partial t_0} f_{1,s} + \frac{\partial}{\partial t_1} f_{0,s}$, etc.

Such a type of request, implies, as is well-known, constraints on the form of $f_{0,s}$ $f_{1,s}$ and $f_{2,s}$, since it can be proven that Eqs. (3) impose $\left(\frac{\partial}{\partial t} f_s\right)_3 \neq 0$, and thus, at most, we may require $\left(\frac{\partial}{\partial t} f_s\right)_i = 0$ for $i = 0, 1, 2$. An important result is, however, that contrary to what occurs in a collisionless magnetoplasma [8], such constraints do not assure, in general, the existence of the same adiabatic invariants occurring in that case.

In order to examine this problem in detail let us define, initially, what is an " adiabatic invariant " in the present context. Here we define a dynamical variable $F_s(\boldsymbol{r}, \boldsymbol{v}, t)$ as an " adiabatic invariant of order $i$ " if results:

$$L_{i,s}(F) = 0 \tag{5}$$

for all species $s$.

Thus, imposing that $f_{0,s}$ be independent of $t_0$ implies, in turn, that $f_{0,s}$ be independent of $\zeta$ (being $\zeta$ the azimuth $\zeta = \operatorname{arctg}(\boldsymbol{v} \cdot \boldsymbol{b} / \boldsymbol{v} \cdot \boldsymbol{p})$, with $(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{n})$ being a right-handed system of unit vectors), since results $\boldsymbol{v} \wedge \boldsymbol{\Omega}_{0,s} \cdot \frac{\partial}{\partial \boldsymbol{v}} f_{0,s} = -\Omega_{0,s} \frac{\partial}{\partial \zeta} f_{0,s} = 0$. We infer that $f_{0,s} = f_{0,s}(\boldsymbol{r}, \xi, \mu, \sigma)$, were $\xi = v^2/2$ and $\sigma = \operatorname{sign}(v_{\|})$, with $v_{\|} = \boldsymbol{v} \cdot \boldsymbol{n}$.

By taking the $\zeta$-average of Eq. (3), for $i = 2$, and introducing the notation $f = \bar{f} + \tilde{f}$, with $\bar{f} = (2\pi)^{-1} \oint d\zeta f$, one obtains:

$$\frac{\partial}{\partial t_0} \bar{f}_{1,s} + \frac{\partial}{\partial t_1} f_{0,s} + v_{\|} \boldsymbol{n} \cdot \nabla f_{0,s} + \frac{e_s}{m_s} \mathbf{E}_0 \cdot \boldsymbol{n} v_{\|} \frac{\partial}{\partial \xi} f_{0,s} = C_s(f_0 \mid f_0) . \tag{6}$$

For times which are bounded by $T_0/\varepsilon_s$, where $T_0 = t_0^{(1)} - t_0^{(0)}$ (being $t_0^{(0)}$ and $t_0^{(1)}$ the " initial " and " final " times) is some time-interval, we find that in order to prevent secular behaviours of $\bar{f}_{1,s}$, one must require:

$$\frac{\partial}{\partial t_1} f_{0,s} + v_{\|} \boldsymbol{n} \cdot \nabla f_{0,s} + \frac{e_s}{m_s} \mathbf{E}_0 \cdot \boldsymbol{n} v_{\|} \frac{\partial}{\partial \xi} f_{0,s} = C_s(f_0 \mid f_0) \tag{7}$$

and hence $\frac{\partial}{\partial t_0} \bar{f}_{1,s} = 0$. Imposing that $f_{0,s}$ be also independent of $t_1$, we obtain finally the " equilibrium " Fokker-Planck equation, well-known in the literature [1]:

$$v_{\|}\, \boldsymbol{n} \cdot \nabla f_{0,s} + \frac{e_s}{m_s} \mathbf{E}_0 \cdot \boldsymbol{n}\, v_{\|} \frac{\partial}{\partial \xi} f_{0,s} = \mathrm{C}_s (f_0 \mid f_0) \tag{8}$$

This equation indicates that, to this order, both for circulating and magnetically trapped particles, $\xi$ , $\mu$ , $\sigma$ are not adiabatic invariants of the first order. On the contrary, by introducing the Euler potentials $\alpha$ , $\beta$ defined in such a way that $\mathbf{B} = \nabla\alpha\Lambda\nabla\beta$ (where, for definitess, $\alpha$ shall be identified with the scalar kinetic pressure $\pi_0 = \Sigma_s \mathrm{N}_{0,s} \mathrm{T}_{0,s}$) we find, as usual, that $\alpha$ and $\beta$ are first-order adiabatic invariants.

On the other hand, if we introduce the following additional assumptions:

$$\mathbf{E}_0 \cdot \boldsymbol{n} = 0 \tag{9a}$$

$$\boldsymbol{n} \cdot \nabla f_{0,s} = 0 \tag{9b}$$

$$\mathrm{C}_s (f_0 \mid f_0) = 0 \tag{9c}$$

$\forall s = 1 , 2 , \cdots , r$ ($r$ denoting the number of particle species), we infer immediately that $f_0$, reduces to a local maxwellian distribution, i.e.:

$$f_{0,s} = f_{\mathrm{M},s}(v) = \mathrm{N}_{0,s} (2\, \pi)^{-3/2}\, v_{th,s}^3 \exp \{- x_s^2\} \tag{10}$$

with $x_s = v/v_{th,s}$ and $v_{th,s} = (2\, \mathrm{T}_{0,s}/m_s)^{\frac{1}{2}}$, where $\boldsymbol{n} \cdot \nabla \mathrm{N}_{0,s} = \boldsymbol{n} \cdot \nabla \mathrm{T}_{0,s} = 0$ and $\mathrm{T}_{0,s} = \mathrm{T}_{0,k}$ for all particle species. Eqs. (9) and (10) imply, in turn, that $\xi$, as well as $\alpha$ and $\beta$, are, in this case, first-order adiabatic invariants, just as for a collisionless magnetoplasma.

This result can be recovered by inspecting the Lagrangian subsidiary equations associated to Eq. (8), written in the form:

$$\frac{\partial}{\partial l} f_{0,s}\, \dot{l} + \frac{\partial}{\partial \xi} f_{0,s}\, \dot{\xi} = \mathrm{C}_s (f_0 \mid f_0) \tag{8'}$$

where we have defined $\dot{l} = \dfrac{\mathrm{d}l}{\mathrm{d}t_1}$ and $\dot{\xi} = \dfrac{\mathrm{d}\xi}{\mathrm{d}t_1}$, with:

$$\begin{aligned} \dot{l} &= v_{\|} \\ \dot{\xi} &= \frac{e_s}{m_s} \mathbf{E}_0 \cdot \boldsymbol{n} v_{\|} \end{aligned} \tag{11}$$

where results $\dot{\xi} = 0$ thanks to Eq. (9b). Here we have introduced a curvilinear coordinate $l$ along a given magnetic flux line. We remark that, on the contrary, the magnetic moment $\mu$ is generally not conserved in the present case; in fact if a solution $f_{0,s}(\alpha, \beta, \xi, \mu)$ is chosen results $C_s(f_0 \mid f_0) \neq 0$, which implies, in validity of Eqs. (9a) and (9b) that $\frac{\partial}{\partial t_1} f_{0,s} \neq 0$.

Thus we find that the existence of first-order adiabatic invariants can be inferred also for a collisional magnetoplasma but under constraints much more severe on the choice of the equilibrium distribution $f_{0,s}$ than in the case of a collisionless system (2).

## 3. Consequences of the condition of spatial symmetry

Let us now assume that the magnetoplasma has a some sort of spatial symmetry, i.e., we shall require that the electro-magnetic fields $\{\mathbf{E}, \mathbf{B}\}$ and the one-particle distribution function $f_s(\boldsymbol{r}, \boldsymbol{v}, t)$ are invariant w.r. to a coordinate transformation $\theta \to \theta + k$ ($\forall k \in \mathrm{R}$), such that $\boldsymbol{n} \cdot \hat{e}_\theta \neq 0$, being $\hat{e}_\theta = \nabla\theta / \nabla\theta \mid$. We denote here by $p_{c,s} = q_s + \frac{e_s}{c} \mathbf{A} \cdot \frac{\partial \boldsymbol{v}}{\partial \dot{\theta}}$ $\left(\text{with } q_s = m_s \boldsymbol{v} \cdot \frac{\partial \boldsymbol{v}}{\partial \dot{\theta}}\right)$ the canonical momentum of a particle of the species $s$ which is conjugated to $\theta$ in terms of the $s$-th particle lagrangian $\mathscr{L} = \mathrm{T} - e_s \Phi + \frac{e_s}{c} \mathbf{A} \cdot \boldsymbol{v}$ (being $\{\mathbf{A}, \Phi\}$ the electromagnetic potentials related to $\{\mathbf{E}, \mathbf{B}\}$, i.e. defined as $p_{c,s} = \partial \mathscr{L} / \partial \dot{\theta}$.

Let us examine what general consequences may be inferred from such an assumption in the context of SLRO. For this purpose it is sufficient here to examine only the first and second order perturbative equations of (3), i.e. for $i = 1, 2$. In particular, as far as concerns the last equation, under the assumptions (9) and after taking a $\zeta$-average, one recovers the usual drift Fokker-Planck equation [1]:

$$v_{\|} \boldsymbol{n} \cdot \nabla \bar{f}_{1,s} + \boldsymbol{v}_{\mathrm{D},s} \cdot \nabla f_{\mathrm{M},s} + \frac{e_s}{m_s} \mathbf{E}_1 \cdot \boldsymbol{n} v_{\|} \frac{\partial}{\partial \xi} f_{\mathrm{M},s} = \mathrm{C}_s(f_0 \mid \bar{f}_1) \tag{12}$$

(obtained requiring $\frac{\partial}{\partial t_2} f_{\mathrm{M},s} = 0$ and imposing, furthermore, $\frac{\partial}{\partial t_1} \bar{f}_{1,s} = \frac{\partial}{\partial t_0} \bar{f}_{2,s} = 0$ in order to avoid secular behaviours in $t_0$ and $t_1$).

(2) In a collisionless plasma, assuming $\mathbf{E}_0 \cdot \boldsymbol{n} = 0$, it is well-known, in fact, that the equilibrium distribution is of the form $f_{0,s} = f_{0,s}(\xi, \mu, \mathrm{J}_{\|})$ for particles which are magnetically trapped (i.e., for which exists a $l_1$ on a given flux line for which $v_{\|}(l_1) = 0$ and where the particle is reflected, changing the sign of $v_{\|}$) and $f_{0,s} = f_{0,s}(\xi, \mu, \mathrm{J}_{\|}, \sigma)$ for particles which are circulating, i.e. non-trapped [8]. Here $\mathrm{J}_{\|} = \oint_{\mathrm{C}} dl v_{\|}$ is the so-called parallel adiabatic invariant which results a function of $\alpha$, $\beta$, $\xi$ and $\mu$. We remark that as for $\mu$, also $\mathrm{J}_{\|}$ is generally not conserved in a collisional system.

We intend to prove that Eq. (12) admits a number of second-order adiabatic invariants; more precisely:

THEOREM 1. *$p_{c,s} f_{M,s}$ is a second-order adiabatic invariant; while* $Q = \Sigma_s \int d^3 v q_s f_s (r, v, t)$ *is an adiabatic invariant of the second order if the plasma is locally quasi-neutral in the sense that:*

$$\Sigma_s e_s N_{0,s} \sim 0(\varepsilon^\delta) \tag{13}$$

*with* $\delta \geqq 2$ *and* $\varepsilon = \max\{\varepsilon_1, \cdots, \varepsilon_r\}$ *(r is the total number of particle species of the plasma.*

The proof for $p_{c,s} f_{M,s}$ can be obtained by verifying directly that Eq. (12) is invariant w.r. to a transformation of the type:

$$f_{1,s} \to f_{1,s}(\alpha_{1,s}) = f_{1,s} + \alpha_{1,s} p_{c,s} f_{M,s} \tag{14}$$

being $\alpha_{1,s} = \alpha_{1,s}(\alpha)$ and $\alpha_{1,s}(\alpha) = \alpha_{1,k}(\alpha)$ for two arbitrary species $s$ and $k$. In fact, since by assumption $p_{c,s}$ is an exact constant of motion for the " unperturbed " system, results $[H, p_{c,s}] = 0$, being $H = T + e_s \Phi$; thus to leading order in $\varepsilon_s$, one obtains, recalling also Eq. (9b):

$$\overline{\left(v \cdot \nabla + v \wedge \Omega_{0,s} \cdot \frac{\partial}{\partial v}\right) p_{c,s} f_{M,}} = \overline{p_{c,s}} v_\| n \cdot \nabla f_{M,s} = 0 . \tag{15}$$

In order to show that results analogously:

$$C_s(f_M \mid \alpha_1 \overline{p}_c f_M) = 0 \tag{16}$$

we distinguish between the two contributions appearing in $p_{c,s}$, i.e. $q_s = m_s v \cdot \frac{\partial v}{\partial \theta}$ and $\frac{e_s}{c} \mathbf{A} \cdot \frac{\partial v}{\partial \theta}$. Thanks to galileian invariance of the Fokker-Planck collision operator and noting that $\frac{\partial v}{\partial \theta}$ is independent of $v$, one infers $C_s(f_M \mid \alpha_1 \overline{q} f_M) = 0$. Similarly, since $\mathbf{A}$ results, to the lowest significant order in $\varepsilon$, independent of $v$, thanks to Eq. (9c) one gets $C_s\left(f_M \mid \alpha_1 \frac{e}{c} \mathbf{A} \cdot \frac{\partial v}{\partial \theta} f_M\right) = 0$.

In order to show that Q results analogously a second-order adiabatic invariant, in validity of (13), one needs to prove that:

$$Q = C(1 + 0(\varepsilon)) \tag{19}$$

being C some exact constant of motion to be appropriately determined.

For this purpose it is sufficient to notice that the average canonical momentum of the plasma, conjugated to $\theta$, which reads:

$$P_c = \Sigma_s \int d^3 v p_{c,s} f_s(\boldsymbol{r}, \boldsymbol{v}, t) \tag{20}$$

is an exact constant of motion, provided $\mathbf{A}$ is independent of $\boldsymbol{v}$. In fact one infers:

$$\frac{d}{dt} P_c = \Sigma_s \int d^3 v \{[p_{c,s}, H] f_s + p_{c,s} C_s(f \mid f)\} \tag{21}$$

and recalling that $[p_{c,s}, H] = 0$, while results:

$$\Sigma_s \int d^3 v p_{c,s} C_s(f \mid f) = \Sigma_s \frac{\partial \boldsymbol{v}}{\partial \theta} \cdot \int d^3 v m_s \boldsymbol{v} C_s(f \mid f) + \tag{22}$$

$$+ \Sigma_s \frac{e_s}{c} \mathbf{A} \cdot \frac{\partial \boldsymbol{v}}{\partial \theta} \int d^3 v C_s(f \mid f) = 0$$

thanks to momentum and particle number conservation, one gets $\frac{d}{dt} p_c = 0$
If we suppose, furthermore that the plasma is "strictly" neutral in the sense:

$$\Sigma_s e_s N_{0,s} = 0 \tag{23}$$

from Eqs. (20) and (23) it follows $Q = C = P_c$.

If, instead, only the weaker quasi-neutrality condition (13) is fulfilled, it is immediate to infer Eq. (19), when noticing—in addition—that:

$$\int d^3 v \boldsymbol{v} f_s(\boldsymbol{r}, \boldsymbol{v}, t) = \int d^3 v \boldsymbol{v} f_{1,s}(\boldsymbol{r}, \boldsymbol{v}, t) \quad (1 + 0(\varepsilon_s)) \sim 0(\varepsilon_s) \tag{24}$$

since, by assumption, the local maxwellian distribution has vanishing mass velocity.

The physical meaning of the adiabatic invariant Q is immediate. In particular, for toroidal axisymmetric equilibria Q is simply the component along the principal axis of the torus of the angular momentum ($Q = L_z = \Sigma_s m R \hat{e}_\theta \cdot \int d^3 v \boldsymbol{v} f_{1,s}(\boldsymbol{r}, \boldsymbol{v}, t)$, where R is the distance from the principal axis of the torus).

In order to generalize the previous results to a magnetoplasma in the presence of non-rigid plasma motions (see I for a discussion of the matter), we consider here an "equilibrium" of the form:

$$f_{0,s}(\boldsymbol{r}, \boldsymbol{v}, t) = f_{M,s}(v)(1 + \alpha_{1,s} p_{c,s}) \tag{25}$$

with $\alpha_{1,s} p_{c,s} \sim 0(\varepsilon_s)$ and letting $\Delta v_{sk} = v_{1,s} - v_{1,k} \neq 0$ in general $\left(\text{here } v_{1,s} = 2\, T_{0,s}\, \alpha_{1,s} \frac{\partial \boldsymbol{v}}{\partial \theta} \cdot \boldsymbol{n}\right)$. In particular we intend to look for particular solutions of Eq. (12) such that:

$$\Delta v_{sk} \neq 0 \tag{26a}$$

$$\frac{\partial}{\partial t_2} f_{0,s} = 0 . \tag{26b}$$

In this case there results, instead of Eq. (16):

$$C_s(f_M \mid \alpha_1 \bar{p} c f_M) = \frac{1}{2\, T_{0,s}} \Sigma_k\, m_s\, \Delta v_{sk}\, q_{sk} \frac{\partial}{\partial \boldsymbol{v}} \cdot \int d^3 v' \frac{\partial^2 u}{\partial \boldsymbol{v}\, \partial \boldsymbol{v}} \cdot \boldsymbol{n} f_{M,s}(v) f_{M,k}(v') \tag{16'}$$

and thus it is apparent that Eq. (12) depends explicitly from $\Delta v_{sk}$, with $k = 1, 2, \cdots r$. Notice that $\Delta v_{sk}$ appears in Eq. (12) as an arbitrary parameter. Hence we infer:

THEOREM 2. *$\Delta v_{sk}$ $(s, k = 1, 2, \cdots r)$ is a second-order adiabatic invariant.*

The value of $\Delta v_{sk}$ is, however, no more arbitrary in the limit $\rho_s \gg 1$ (here $\rho_s$ is defined as $\rho_s = \nu_{s,\mathrm{eff}}/\omega_b$, where $\nu_{s,eff}$ is the effective collision frequency and $\omega_b$ the bounce or transit frequency characterizing the unperturbed particle motion along a magnetic flux line). In this limit (called "strongly collisional limit"), by introducing a perturbative expansion in terms of $1/\rho_s$ one obtains to leading order (in $1/\rho_s$) the equation:

$$C_s(f_M \mid f_1(\alpha_1)) = 0 \tag{27}$$

which delivers $\Delta v_{sk} = 0$. (for $s, k = 1, 2, \cdots r$).

From the previous analys's we obtain the 'mportant conclusion that for spatially symmetric magnetoplasmas, which are locally quasi-neutral in the sense (13), the plasma states correspond to definite values of the mechanical momentum Q and of the relative velocity $\Delta v_{sk}$. Since such dynamical variables are at least second-order adiabatic invariants, we conclude that they vary on a time-scale comparable with (or possibly even larger than) the classical or neoclassical collisional diffusion time scales [4].

## List of symbols

| | |
|---|---|
| $\mathrm{A}$ | vector potential; |
| $\alpha$ , $\beta$ | Euler potentials; |
| $\mathbf{B} = \nabla\alpha \wedge \nabla\beta$ | magnetic field; |
| $\mathbf{E}$ | electric field; |
| $e_s$ | electric charge; |
| $f_s(\boldsymbol{r}, \boldsymbol{v}, t)$ | one-particle distribution function of the $s$-th species; |
| $f_{0,s}(\boldsymbol{r}, \boldsymbol{v})$ | equilibrium distribution function; |
| $f_{1,s}(\boldsymbol{r}, \boldsymbol{v})$ | first-order perturbation of the distribution function; |
| $\phi$ | electrostatic potential; |
| $\mathrm{J}_{\|}$ | parallel adiabatic invariant; |
| $\mathrm{L}$ | characteristic scale length of the equilibrium; |
| $m_s$ | mass; |
| $\mu$ | magnetic moment per unit mass; |
| $\boldsymbol{n} = \mathbf{B}/\mathrm{B}$ | |
| $\mathrm{N}_s$ | number density; |
| $\pi_0$ | scalar kinetic pressure; |
| $p_{c,s}$ | canonical momentum of a particle of the $s$-th species, conjugated to $\theta$; |
| $\mathrm{P}_c$ | canonical momentum of the plasma; |
| $\mathrm{Q}$ | angular momentum of the plasma; |
| $r_s$ | Larmor radius; |
| $\mathrm{T}_s$ | temperature; |
| $\boldsymbol{v}_{\mathrm{D},s} = \boldsymbol{n} \wedge (\mu\nabla\mathrm{B} + v_{\|}^2\, \boldsymbol{n} \cdot \nabla\boldsymbol{n})/\Omega_s$ | diamagnetic drift velocity; |
| $\xi$ | kinetic energy per unit mass; |
| $\theta$ | cyclic coordinate; |
| $\Omega_s$ | Larmor frequency. |

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