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On the Cauchy problem in linear viscoelasticity

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Fisica matematica. — *On the Cauchy problem in linear viscoelasticity* (*). Nota di PASQUALE RENNO, presentata (**) dal Socio D. GRAFFI.

RIASSUNTO. — Con riferimento all'operatore integrodifferenziale della viscoelasticità lineare nella formulazione *creep*, si determina la soluzione fondamentale E in corrispondenza di un'arbitraria funzione di memoria. Di conseguenza viene risolto esplicitamente il problema di Cauchy relativo al moto unidimensionale di un sistema viscoelastico $\mathscr{B}$, omogeneo ed isotropo, determinato da dati iniziali e storia di stress comunque prefissati. Successivamente, nell'ambito di opportune ipotesi di memoria labile, si dimostrano alcune proprietà di E che consentono rigorose approssimazioni del moto di $\mathscr{B}$ in un intervallo finito di tempo.

0. Let $\mathscr{B}$ be an isotropic, intrinsically homogeneous body with a *linearly viscoelastic behaviour of creep type* and let $\boldsymbol{u}$ denote the displacement from an undeformed homogeneous reference configuration $\mathscr{R}$. According to the translation invariant axiom, the well known creep representation of the linear one-dimensional motions of $\mathscr{B}$ is given by

$$\mathrm{L}u = c^2 u_{xx} - u_{tt} - \int_0^t g(t-\tau)\, u_{\tau\tau}(x,\tau)\, \mathrm{d}\tau = -\tilde{f}\,, \tag{0.1}$$

where $u(x,t)$ is the single non-vanishing displacement component and $\tilde{f}$ represents a source term depending on the body forces and on the prescribed *past history of stress*. Further, the response function $g(t)$ denotes the ratio $\dot{\mathrm{J}}(t)/\mathrm{J}(0)$, where $\mathrm{J}(t)$ is the *creep compliance* (Sect. 1).

As it is well known, numerous and various problems related to (0.1) and to the corresponding relaxation representation have been the subject of several and interesting researches (e.g. [3, . . . , 8] and [12]). So, as for the Cauchy problem, J. Barberan and I. Herrera in [5, 6] have established the existence and the uniqueness of the solution also for inhomogeneous media. As for the signalling problem, D. Graffi [3] and M. Fabrizio [7] have constructed the explicit solution by means of series of transcendental functions. Further, the case

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of a rod of finite length with both ends fixed has been considered in [8] by C. Dafermos, who has investigated the asymptotic stability of solutions of an abstract integrodifferential equation in the framework of the theory of dynamical systems. The solution is asymptotically stable when $g(t)$, besides appropriate hypotheses of fading memory, verifies also a convexity assumption. This problem of the viscoelastic rod has been analyzed also in [12] by D. Graffi who has proved that when $g(t)$ is approximated by a Prony series, the eigenoscillations of $\mathscr{B}$ result by a combination of a damped oscillation and aperiodic motions.

The special case of a *standard linear solid* $\mathscr{B}_s$ has been widely discussed. In [16], when $x \in \mathrm{R}^n$ $(n = 1, 2, 3)$, the explicit fundamental solution related to $\mathscr{B}_s$, whatever the number $n = 1, 2, 3$ of space dimensions may be, has been constructed. Moreover various basic aspects of the wave behaviour such as diffusion, singular perturbation problems and asymptotic properties have been evaluated. Further, maximum principles necessary to solve for $\mathscr{B}_s$ also *unilateral problems* [15] have been established.

In this paper—in order to extend to the case of $\mathscr{B}$ the analysis stated in [16] for $\mathscr{B}_s$—the fundamental solution $<\mathrm{E}, \chi>$ $(\chi \in \mathscr{S}(\mathrm{R}^2))$ of the operator L at first is constructed (Sect. 2). Consequently, the explicit solution of the Cauchy problem in all of the space and with quite arbitrary data is achieved (Sect. 3).

Successively, some properties of $<\mathrm{E}, \chi>$ connected with meaningful and usual hypotheses of fading memory for $g(t)$ are investigated (Sect. 4). So, when $g(t) > 0$, $\dot{g}(t) < 0$ on R^+, the fundamental solution of L is a *tempered distribution associated with a never negative function* $\mathrm{E}(x, t)$ which can be estimated in terms of the well known fundamental solution $\mathscr{H}[g_0]$ related to the case $g(t) = \text{const.} = g_0 > 0$:

$$\mathscr{K}[g_0] = (2c)^{-1}\, \eta(t - r)\, e^{-g_0 t/2}\, \mathrm{I}_0\left(\tfrac{1}{2} g_0 \sqrt{t^2 - r^2}\right), \tag{0.2}$$

where $r = |x|/c$, $\eta(t)$ is the Heaviside function and I_0 is the modified Bessel function of the first kind.

In fact, if Γ denotes the support of E that is the forward characteristic cone related to L, the following theorem holds.

THEOREM 0.1. *When $g(t)$ is a strictly positive $\mathrm{C}^2(\mathrm{R}^+)$ function with $\dot{g}(t) < 0$ on R^+, the fundamental solution of the operator* L *is a tempered distribution of order zero induced by a never negative $\mathrm{C}^2(\mathring{\Gamma})$ function* E. *Moreover, everywhere in $\mathring{\Gamma}$,* E *satisfies the estimates*

$$0 < \mathscr{K}[g(0)] \leq \mathrm{E} \leq \mathscr{K}[g(0)] + k(t^2 - r^2)\, \mathscr{K}[g(t)], \tag{0.3}$$

where $\mathscr{H}[\cdot]$ is defined in (0.2), $k = \sup_t [-\dot{g}(t)]$ *and the equality holds iff $r = t$.*

So, on every bounded initial time interval $[0, T]$, *the behaviour of* $\mathcal{B}$ *can be rigorously compared with that of the media* $\mathcal{B}_0$ *and* $\mathcal{B}_T$ *characterized by the constant memories* $g(0)$ *and* $g(T)$. In fact, by (0.3) one has

$$\mathcal{X}[g(0)] \leq E \leq (1 + kT^2)\,\mathcal{X}[g(T)] \tag{0.4}$$

as $\mathcal{H}$ is a decreasing function of g. Further, for any arbitrary finite t, the inequality (0.3) permits us to estimate the solution u of the initial value problem related to (0.1) in terms of the data. Obviously, when t is large, the problem of the asymptotic behaviour must be investigated. But, as Dafermos has proved even when $x \in [0, 1]$, more restrictive conditions on $g(t)$ and the da a must be requested. This analysis, together with applications of (0.3), will be dealt with successively.

1. In the framework of the onedimensional linear theory of elasticity let $u(x, t)$, $\sigma(x, t)$, $e(x, t)$ be the single nonvanishing components of the displacement field, the stress tensor and the strain tensor respectively. Further, let $f(x, t)$ be a volume density of prescribed forces and let ρ denote the constant mass density in the homogeneous reference configuration $\mathcal{R}$. When one assumes for $\mathcal{B}$ an *elastic heredity of creep type*, the onedimensional motions of $\mathcal{B}$ are described by the well known equations.

$$\rho u_{tt} = \sigma_x + f \;, \quad e = u_x \tag{1.1}$$

$$e = J(0)\left[\sigma(t) + \int_{-\infty}^{t} g(t-\tau)\,\sigma(\tau)\,d\tau\right], \tag{1.2}$$

where $J(0)$ is the initial value of the creep compliance $J(t)$ and $g(t) = \dot{J}(t)/J(0)$. By (1.1)-(1.2) obviously one deduces (0.1) with $c^2 = [\rho J(0)]^{-1}$ and

$$\tilde{f} = \rho^{-1} f + \rho^{-1} \int_0^t g(t-\tau) f(x, \tau)\,d\tau + f_\sigma \tag{1.3}$$

where

$$f_\sigma = -\rho^{-1} \int_{-\infty}^{0} g(t-\tau)\,\sigma_x(x, \tau)\,d\tau \;. \tag{1.4}$$

Of course, according to Volterra, it is assumed that the past history of $\sigma(x, \tau)$ is prescribed on $R \times (-\infty, 0]$ and is such that the integral in (1.4) is meaningful and the derivation under the integral sign with respect to the variable space is feasible.

Then, referring to the half-space

$$Y_+^2 = \{(x , t) : x \in R , t > 0\} , \tag{1.5}$$

the classic initial-value problem $\mathscr{P}$ related to (0.1) can be given the form

$$Lu = -\tilde{f} \qquad (x , t) \in Y_+^2 \tag{1.6}$$

$$\partial_t^i u (x , 0^+) = f_i (x) \qquad (i = 0 , 1) \quad x \in R . \tag{1.7}$$

As for the response function $g(t)$, we will assume that g is of the class $C^2 (R^+)$ and is compatible with the fading memory axiom [2 , 10 , . . . 13]. The special case of Th. 0.1 will be considered in Sect. 4.

2. Within the limits of an *heuristic* research of the solution of the Cauchy problem $\mathscr{P}$, we apply to (1.6)-(1.7) the Fourier operator with respect to x and he Laplace operator $\mathscr{L}$ with respect to t. Let s be the parameter of the $\mathscr{L}_t$ transformation and let

$$\hat{v} (x , s) = \mathscr{L}_t \, v (x , t) \ , \ v_1 (x) \cdot v_2 (x) = \int_R v_1 (\xi) \, v_2 (x - \xi) \, d\xi \ . \tag{2.1}$$

Then, by means of standard computations, by (1.6)-(1.7) it follows

$$\hat{u} (x , s) = \hat{E} (x , s) \cdot [\hat{f}_\sigma + (1 + \hat{g} (s)) (sf_0 + f_1 + \rho^{-1} \hat{f})] \ , \tag{2.2}$$

where $\hat{E}$ is the $\mathscr{L}_t$-transform formally defined by the symbolic relation

$$\hat{E} (x , s) = [2cs \sqrt{1 + \hat{g} (s)}]^{-1} \exp \left[- \frac{|x|}{c} s \sqrt{1 + \hat{g} (s)} \right] . \tag{2.3}$$

Obviously, when $E = \mathscr{L}^{-1} \hat{E}$ is known, the inverse transform of the other term $(1 + \hat{g}) \hat{E}$ which appears in (2.2) is given by

$$E_1 (x , t) = \mathscr{L}^{-1} [(1 + \hat{g}) \hat{E}] = E + g * E \ , \tag{2.4}$$

where * denotes the convolution on t.

The invers on of $\mathscr{L}$-transforms such as (2.3) has been achieved already by by Graffi in [3] and Fabrizio in [7]. But some properties of E, which we will state in Section 4, do not appear easily deducible by these results.

For this we are going to establish a new inversion formula of (2.3).

Let us consider the Laplace integral ([14] p. 200)

$$(2.5) \qquad \int_b^\infty e^{-pu}\, \mathrm{I}_0\,(a\sqrt{u^2-b^2})\, du = \frac{e^{-b\sqrt{p^2-a^2}}}{\sqrt{p^2-a^2}} \qquad \begin{array}{l} b>0 \\ Re\,(p) > |\,\mathrm{Re}\,(a)\,| \end{array}$$

and let

$$b = r = c^{-1}\,|\,x\,| \quad , \quad p = s\,(1+\hat{g}/2) \quad , \quad a = s\hat{g}/2 \ .$$

Then one has

$$(2.6) \qquad \hat{\mathrm{E}}\,(x\,,\,s) = (2c)^{-1}\int_r^\infty e^{-(1+\hat{g}/2)\,su}\ \mathrm{I}_0\ [(\tfrac{1}{2})\,s\hat{g}\ \sqrt{u^2-r^2}]\ du \ .$$

Further, being

$$(2.7) \qquad \mathrm{I}_0\,(v) = \frac{1}{\pi}\int_0^\pi e^{v\cos\theta}\, d\theta$$

by (2.6)-(2.7) it follows

$$(2.8) \qquad \hat{\mathrm{E}}\,(x\,,\,s) = (2c\pi)^{-1}\int_0^\pi d\theta \int_r^\infty e^{-su-zs\hat{g}}\, du$$

with $z = 2^{-1}\,(u - \cos\theta\,\sqrt{u^2-r^2})$.

If one puts

$$h\,(t) = -\dot{g}\,(t) \quad , \quad g\,(0) = g_0 \quad , \quad \hat{h}\,(s) = \mathscr{L}\,[-\dot{g}\,(t)] = -s\hat{g} + g_0$$

the formula (2.8) becomes

$$(2.9) \qquad \hat{\mathrm{E}}\,(x\,,\,s) = (2\pi c)^{-1}\int_0^\pi d\theta \int_r^\infty e^{-su-zg_0+z\hat{h}(s)}\, du \ .$$

Now, formally one has

$$(2.10) \qquad e^{z\hat{h}(s)} = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!}\,\mathscr{L}\,[\mathrm{G}_n\,(t)]$$

where

$$(2.11) \qquad \mathrm{G}_1\,(t) = -\dot{g} \quad , \quad \mathrm{G}_n\,(t) = -\dot{g} * \mathrm{G}_{n-1} \qquad (n \geq 2)\,.$$

Further, it results

$$(2.12)\qquad (2\pi c)^{-1}\int_0^{\pi} d\theta \int_r^{\infty} e^{-su-rg_0} du = \frac{e^{-r\sqrt{s^2+g_0 s}}}{\sqrt{s^2+g_0 s}} = \mathcal{L}\mathcal{H}\,[g(0)]$$

where $\mathcal{H}$ is the function defined by (0.2). Consequently, by (2.9)-(2.11)-(2.12) easily it follows that

$$(2.13)\qquad \mathrm{E}(x,t) = (2c)^{-1}\eta(t-|x|/c)\ \mathrm{F}(|x|/c,t)$$

where $\mathrm{F}(r,t)$ is

$$(2.14)\qquad \mathrm{F}(r,t) = e^{-g_0 t/2}\,\mathrm{I}_0\left(\tfrac{1}{2}g_0\sqrt{t^2-r^2}\right) + \pi^{-1}\sum_{n=1}^{\infty}\int_0^{\pi} d\theta\int_r^{t} e^{-g_0 z}\frac{z^n}{n!}\,\mathrm{G}_n(t-u)\,du\,.$$

At last, by (2.4) one has

$$\mathrm{E}_1(x,t) = (2c)^{-1}\eta(t-|x|/c)\,\mathrm{F}_1(|x|/c,t)$$

with

$$(2.15)\qquad \mathrm{F}_1(r,t) = \mathrm{F}(r,t) + \int_r^{t} g(t-\tau)\,\mathrm{F}(r,\tau)\,d\tau\,.$$

3. In order to prove rigorously that the symbolic relations now applied allow us to solve explicitly the Cauchy problem $\mathcal{P}$, now we will demonstrate the following theorem. If $\Gamma = \{(x,t) : t > 0,\ |x| \le ct\}$ is the forward characteristic cone, one has:

THEOREM 3.1. *Let* $\dot{g}(t)$ *and* $g(t)$ *be bounded functions on* R^+. *Then the kernel* F *given by* (2.14) *is a* $\mathrm{C}^2(\Gamma)$ *function which defines a regular solution of* (0.1) *with* $\tilde{f} = 0$. *Furthermore, the Laplace integral* $\mathcal{L}_t\,\eta(t-r)\,\mathrm{F}(r,t)$ *converges absolutely in the half-plane* $\mathrm{Re}(s) > 0$ *and one has*

$$(3.1)\qquad \mathcal{L}\eta(t-|x|/c)\,\mathrm{F}(|x|/c,t) = 2c\,\hat{\mathrm{E}}(x,s)$$

with $\hat{\mathrm{E}}$ *defined by* (2.3).

Proof. It is

$$\int_r^t e^{-g_0 z} z^n G_n(t-u)\,du = \int_0^{t-r} e^{-g_0 y} y^n G_n(u)\,du\,.$$

with $y = \frac{1}{2}\,[t-u-\cos\theta\sqrt{(t-u)^2-r^2}]$ and $r = |x|/c$. So the first and the second order derivatives with respect to x and t involve only G_n and $\dot{G}_n$. Now, as $|g(u)| \le M$ for all $u \ge 0$, by (2.11) it is

$$|G_n(u)| \le M^n u^{n-1}/(n-1)! \qquad n \ge 1 \tag{3.2}$$

hence

$$\sum_{n=1}^{\infty} \frac{y^n}{n!}\,|G_n(u)| \le \sum_{n=0}^{\infty} \frac{(My)^{n+1} u^n}{n!\,(n+1)!} = \sqrt{\frac{My}{u}}\; I_1(2\sqrt{Myu})\,.$$

Thus, for all $t \le T$ ($T > 0$ but arbitrary), the series is uniformly convergent. As also $\ddot{g}$ is bounded, for $\dot{G}_n$ inequalities such as (3.2) hold too. Consequently F is a $C^2(\Gamma)$ function such that

$$|F| \le e^{-g_0 t/2}\, I_0(\tfrac{1}{2} g_0\sqrt{t^2-r^2}) + \pi^{-1}\int_0^{\pi} d\theta \int_0^{t-r} e^{-g_0 y}\sqrt{\frac{My}{u}}\; I_1(2\sqrt{Myu})\,du$$

and this implies, by means of the Fubini-Tonelli theorem, that the Laplace integral $\mathscr{L}_t\,\eta(t-r)\,F(r,t)$ converges absolutely for $\mathrm{Re}(s) > 0$. Then, formula (3.1) is an easy consequence of the computations stated in Sect. 2. These properties of F enable us to make rigorous the formal procedure so far employed, proving so that F is a classic solution in Γ of (0.1) (with $\bar{f} = 0$).

The construction of the fundamental solution achieves, as it is well known, the explicit solution of the Cauchy problem with arbitrary data. If one puts

$$u_{f_i} = (2c)^{-1}\int_{x-ct}^{x+ct} f_i(y)\,F_1(|y-x|/c\,,t)\,dy \qquad (i=0\,,1) \tag{3.3}$$

$$u_{\bar{f}} = (2c)^{-1}\int_0^t d\tau \int_{x-c\tau}^{x+c\tau} \bar{f}(y\,,t-\tau)\,F(|y-x|/c\,,\tau)\,dy \tag{3.4}$$

one has:

THEOREM 3.2. *If the initial data* $f_i\,(i=0\,,1)$ *are* $C^{2-i}(R)$ *and the source therm* $\tilde{f}$ *is* $C^1(Y^2_+)$, *then the initial value problem* (1.6)-(1.7) *admits the unique* $C^2(Y^2_+)$ *solution*

$$u = \partial_t u_{f_0} + u_{f_1} + u_{\tilde{f}} \;. \tag{3.5}$$

Proof. As for the heuristic determination of formula (3.5) we refer to uhe symbolic relation (2.2) observing that, owing to $\eta\,(t - |x|/c)$, the convolutions with respect to x are given by (3.3)-(3.4). Further, when $t = |x|/c$ it is

$$F(t\,,t) = F_1(t\,,t) = e^{-g_0 t/2} \tag{3.6}$$

and easily one can see that

$$\mathscr{L}^{-1}\, s\hat{E}\,(x\,,t)\cdot f_0(x) = \partial_t u_f \;.$$

Now, in order to prove that u is a $C^2(Y^2_+)$ solution of $\mathscr{P}$, it suffices to observe that, on the basis of the properties of the kernel F (see Th. 3.1) and the hypotheses on the data, u satisfies everywhere in Y^2_+ the equation (0.1) and the nitial conditions (1.7). The uniqueness is a consequence of well known theorems of Barberan and Herrera [5].

4. The behaviour of the kernel F depends clearly on the properties of the creep compliance J and therefore of the response function $g(t)$. A first result refers to the hypothesis of Theorem 0.1, i.e. $g(t) > 0$ with $\dot{g}(t) < 0$ on R^+. As is well known, this case is meaningful from a physical point of view for the following reasons.

The function $g(t)$ is the conjugate of the memory function $\psi(t)$ related to the relaxation representation

$$\sigma = G(0)\left[e(t) + \int_{-\infty}^{t} \psi(t-\tau)\, e(\tau)\, d\tau\right], \tag{4.1}$$

where $\psi(t) = \dot{G}(t)/G(0)$ and $G(t)$ is the relaxation modulus. Now, well known inversion theorems for strong response functions [1, 4, 10, 13] generally impose precise connections between the properties of ψ and g. Thus, for example, is possible to prove by means of the energy dissipation [12, 13] that when $|\psi|$ is a monotone decreasing function on R^+ vanishing as $t \to \infty$, then ψ must be necessarily negative. This fact implies, according to a Theorem of Volterra, that the conjugate kernel $g(t)$ is positive. Moreover Volterra has also proved the existence of conjugate hereditary coefficients both decreasing in absolute value and vanishing when $t \to \infty$ (see [1], pp.190-194)

Proof of Theorem 0.1. In (2.14) let $F = A' + A''$ be with

$$A' = e^{-g_0 t/2}\, I_0\left(\tfrac{1}{2} g_0 \sqrt{t^2 - r^2}\right) > 0 \tag{4.2}$$

$$A'' = F - A' = \pi^{-1} \int_0^{\pi} d\theta \int_r^t e^{-g_0 z} \sum_{n=1}^{\infty} \frac{z^n}{n!}\, G_n(t-u)\, du\,. \tag{4.3}$$

Now, the hypothesis implies that for all $t > 0$ it is $G_1 = -g > 0$, $G_n = -g * G_{n-1} > 0$ $(n \geq 2)$ and so $A'' \geq 0$, the equality holding iff $r = t$. Further, if $k = \sup_t (-g)$, one easily has

$$G_1 \leq k \quad , \quad G_n \leq k\,[g(0) - g(t)]^{n-1} \qquad (n \geq 1) \tag{4.4}$$

and consequently by (4.3)

$$0 \leq A'' < k\pi^{-1} \int_0^{\pi} d\theta \int_r^t \frac{e^{-zg(t-u)} - e^{-zg(0)}}{g(0) - g(t-u)}\, du\,, \tag{4.5}$$

where the function under the integral sign is clearly summable.

A more crude, but more explicit inequality can be deduced by (4.5) observing that $1 - e^{-y} \leq y$ $(y \geq 0)$ and that $g(t-u) \geq g(t)$. So

$$A'' \leq k\pi^{-1} \int_0^{\pi} d\theta \int_r^t z e^{-zg(t)}\, du\,. \tag{4.6}$$

But, whateve. the parameter a may be, one has

$$\pi^{-1} \int_0^{\pi} e^{-az} z\, d\theta = -\partial_a \left[e^{-au/2}\, I_0\left(\tfrac{1}{2} a\omega\right)\right] = a^{-1} \partial_u \left[e^{-au/2}\, \omega I_1\left(\tfrac{1}{2} a\omega\right)\right]$$

with $\omega = \sqrt{u^2 - r^2}$. So by (4.6), being $a = g(t)$ and $I_1(v) < v I_0(v)$, one deduces

$$A'' \leq k g^{-1} \left[e^{-ug(t)/2}\, \omega I_1\left(\tfrac{1}{2} a\omega\right)\right]_{u=r}^{u=t} \leq k e^{-tg(t)/2} (t^2 - r^2)\, I_0\left[\tfrac{1}{2} g(t) \sqrt{t^2 - r^2}\right]$$

hence (0.3) follows. At last, as $I_0(z) < e^{|z|}$, by (0.3) one deduces **that** $0 < F < 1 + kt^2$ proving so that E *induces a positive temperate distribution of order zero*. So, the proof is complete.

Remark 4.1. As an example, observe that when $\mathscr{B}$ reduces itself to the standard solid $\mathscr{B}_s$ the sum of series in (4.3) is known. In fact, being $g(t) = g_0 e^{-\beta t} (g_0 > 0, \beta > 0)$, one has

$$G_n(t) = g_0^n \beta^n e^{-\beta t} \frac{t^{n-1}}{(n-1)!} \qquad (n \geq 1)$$

and so

$$\sum_{n=1}^{\infty} \frac{z^n}{n!} G_n(t-u) = e^{-\beta(t-u)} \left(\frac{g_0 \beta z}{t-u}\right)^{1/2} I_1 [2 \sqrt{g_0 \beta z (t-u)}] . \tag{4.7}$$

Other applications will be discussed successively.

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