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Some properties of perfect metric spaces

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Topologia. — *Some properties of perfect metric spaces.* Nota di ANGELO BELLA e BIAGIO RICCERI, presentata (*) dal Socio G. SCORZA DRAGONI.

RIASSUNTO. — In questa Nota, dati uno spazio metrico perfetto X ed un suo sottoinsieme K chiuso e raro, si dimostra l'esistenza di una funzione continua $f : X \to [0, 1]$ tale che $\text{int}(f^{-1}(t)) = \emptyset$ per ogni $t \in [0, 1]$, $f(x) = 0$ per ogni $x \in K$ e $f(y) = 1$ per qualche $y \in X \setminus K$. In particolare, ciò permette di dare risposta simultaneamente a due questioni poste in [2]. Si mettono in evidenza, poi, ulteriori conseguenze di tale risultato.

The aim of this Note is to prove Theorem 1 below and to point out some of its consequences.

THEOREM 1. *Let* X *be a perfect metric space and* K *a closed and rare subset of* X. *Then, there exists a continuous function* $f: X \to [0, 1]$ *such that:*

(1) $\text{int}(f^{-1}(t)) = \emptyset$ *for every* $t \in [0, 1]$;

(2) $f(x) = 0$ *for every* $x \in K$;

(3) $\{0, 1\} \subseteq f(X)$.

Proof. Let $\mathcal{B}$ be a σ-discrete base of X, so that $\mathcal{B} = \bigcup_{n \in N} \mathcal{B}_n$, where each $\mathcal{B}_n$ is a discrete family of open subsets of X (see [1], p. 127). We now construct a sequence $\{D_n\}$ of subsets of X and a sequence $\{f_n\}$ of real functions on X, having, for every $n \in \mathbf{N}$, the following properties:

$(1)_n$ the set D_n is discrete and, if $n \geq 2$, $D_n \cap \left(\bigcup_{i=1}^{n-1} D_i\right) = \emptyset$;

$(2)_n$ for every $B^{(n)} \in \mathcal{B}_n$, the set $D_n \cap (B^{(n)} \setminus K)$ has exactly two distinct points $x_{B^{(n)}}, y_{B^{(n)}}$;

$(3)_n$ the function f_n is continuous, $f_n(X) \subseteq [0, 1]$, $f_n(x) = 0$ for every $x \in K$, $f_n(x_{B^{(i)}}) = 1$ and $f_n(y_{B^{(i)}}) = 0$ for every $i = 1, \cdots, n$ and every $B^{(i)} \in \mathcal{B}_i$;

$(4)_n$ if $n \geq 2$, one has $\sum_{i=1}^{n-1} \left| f_i(x_{B^{(n)}}) - f_i(y_{B^{(n)}}) \right| < \frac{1}{2^n}$ for every $B^{(n)} \in \mathcal{B}_n$.

(*) Nella seduta del 26 novembre 1983.

We proceed by induction on n. First, for every $B^{(1)} \in \mathscr{B}_1$, choose in $B^{(1)} \setminus K$ two distinct points $x_{B^{(1)}}, y_{B^{(1)}}$. This is possible since the set $B^{(1)} \setminus K$ is non-empty and open and there is no isolated point in X.

Put

$$D_1 = \bigcup_{B^{(1)} \in \mathscr{B}_1} \{x_{B^{(1)}}, y_{B^{(1)}}\} .$$

Since the family $\mathscr{B}_1$ is discrete, the set D_1 is also discrete. Now consider the function $g_1 : D_1 \cup K \to \mathbf{R}$ defined by

$$g_1(x) = \begin{cases} 1 & \text{if } x = x_{B^{(1)}} , \; B^{(1)} \in \mathscr{B}_1 \\ 0 & \text{if } x \in K \text{ or } x = y_{B^{(1)}} , \; B^{(1)} \in \mathscr{B}_1 . \end{cases}$$

Clearly, the set $D_1 \cup K$ is closed and the function g_1 is continuous. Then, by the Tietze extension theorem, there exists a continuous function $f_1 : X \to [0, 1]$ such that $f_{1|D_1 \cup K} = g_1$. Thus, the first stage of our construction is complete. Now, suppose that sets $D_1, \cdots, D_n$ and functions $f_1, \cdots, f_n$ satisfying $(1)_i$ - $(3)_i$, $i = 1, \cdots, n$, have been constructed. Let us show that it is possible to define a set D_{n+1} and a function f_{n+1} satisfying $(1)_{n+1}$ - $(4)_{n+1}$. Since the functions f_i, $i = 1 \cdots, n$, are continuous, for every $B^{n+1} \in \mathscr{B}_{n+1}$, we can choose in $B^{(n+1)} \setminus \left(\bigcup_{i=1}^{n} D_i \cup K \right)$ two distinct points $x_{B^{(n+1)}}, y_{B^{(n+1)}}$ such that

$$\sum_{i=1}^{n} | f_i(x_{B^{(n+1)}}) - f_i(y_{B^{(n+1)}}) | < \frac{1}{2^{n+1}} .$$

Put

$$D_{n+1} = \bigcup_{B^{(n+1)} \in \mathscr{B}_{n+1}} \{x_{B^{(n+1)}} , y_{B^{(n+1)}}\} .$$

Let $g_{n+1} : \bigcup_{i=1}^{n+1} D_i \cup K \to \mathbf{R}$ be the function defined as follows:

$$g_{n+1}(x) = \begin{cases} 1 & \text{if } x = x_{B^{(i)}} , \; B^{(i)} \in \mathscr{B}_i , \; i = 1, \cdots, n+1 \\ 0 & \text{if } x \in K \text{ or } x = y_{B^{(i)}} , \; B^{(i)} \in \mathscr{B}_i , \; i = 1, \cdots, n+1 . \end{cases}$$

As the set $\bigcup_{i=1}^{n+1} D_i \cup K$ is closed and the function g_{n+1} is continuous, there exists a continuous function $f_{n+1} : X \to [0, 1]$ such that $f_{n+1|\bigcup_{i=1}^{n+1} D_i \cup K} = g_{n+1}$.

Of course, the set D_{n+1} and the function f_{n+1} satisfy $(1)_{n+1}-(4)_{n+1}$. Now, or every $x \in X$, put

$$f(x)=\sum_{n=1}^{\infty} \frac{f_n(x)}{2^n}.$$

The function f is continuous, $f(X) \subseteq [0,1]$ and $f(x)=0$ for every $x \in K$. Moreover, if $B^{(1)} \in \mathscr{B}_1$, we have $f(x_{B^{(1)}})=1$ and $f(y_{B^{(1)}})=0$.

Finally, let us show that $\operatorname{int}(f^{-1}(t))=\varnothing$ for every $t \in [0,1]$. Suppose, on the contrary, that there exists a non-empty open set $\Omega \subseteq X$ such that $f_{|\Omega}$ is constant. Let $\bar{n} \in \mathbf{N}$ and $B^{(\bar{n})} \in \mathscr{B}_{\bar{n}}$ be such that $B^{(\bar{n})} \subseteq \Omega$. By a preceding remark, it must be $\bar{n} \geq 2$. Then, we have

$$0=|f(x_{B^{(\bar{n})}})-f(y_{B^{(\bar{n})}})|=$$

$$=\left|\sum_{i=1}^{\bar{n}-1} \frac{f_i(x_{B^{(\bar{n})}})-f_i(y_{B^{(\bar{n})}})}{2^i}+\sum_{i=\bar{n}}^{\infty} \frac{f_i(x_{B^{(\bar{n})}})-f_i(y_{B^{(\bar{n})}})}{2^i}\right| \geq$$

$$\geq \sum_{i=\bar{n}}^{\infty} \frac{1}{2^i}-\sum_{i=1}^{\bar{n}-1} |f_i(x_{B^{(\bar{n})}})-f_i(y_{B^{(\bar{n})}})| > \frac{1}{2^{\bar{n}-1}}-\frac{1}{2^{\bar{n}}}.$$

That is $1>2$, a contradiction. Thus, our theorem is completely proved.

Remark 1. Theorem 1 gives a negative answer to Problem 3.2 of [2]. Now, we present some consequences of Theorem 1.

THEOREM 2. *Let* X *be a perfect metric space and* $(Y, \|\cdot\|)$ *a normed space. Then, for every continuous function* $f: X \to Y$ *and every* $\varepsilon > 0$, *there exists a continuous function* $f_\varepsilon : X \to Y$ *such that*

(1) $\operatorname{int}(f_\varepsilon^{-1}(y))=\varnothing$ *for every* $y \in Y$;

(2) $\|f_\varepsilon(x)-f(x)\| \leq \varepsilon$ *for every* $x \in X$;

(3) $f_\varepsilon(X) \subseteq \operatorname{conv}(f(X))$, *provided that* f *is non-constant.*

Proof. Put $\tilde{Y}=\{y \in f(X) : \operatorname{int}(f^{-1}(y)) \neq \varnothing\}$ and $A_y=\operatorname{int}(f^{-1}(y))$ for every $y \in Y$. Let K be the boundary of the open set $\bigcup_{y \in \tilde{Y}} A_y$. Plainly, the set K is closed and rare. Then, by Theorem 1, there exists a continuous function $\varphi: X \to [0,1]$ such that $\operatorname{int}(\varphi^{-1}(t))=\varnothing$ for every $t \in [0,1]$ and $\varphi(x)=0$ for every $x \in K$. We may assume $\varepsilon<1$. If the function f is constant, choose $\bar{y} \in Y$, with $\|\bar{y}\|=1$, and, for every $x \in X$, put

$$f_\varepsilon(x)=f(x)+\varepsilon\varphi(x)\bar{y}.$$

If, on the contrary, f is non-constant, choose $y', y'' \in f(X)$, with $y' \neq y''$, and, for every $x \in X$, put

$$f_\varepsilon(x) = \begin{cases} f(x) + \dfrac{\varepsilon\varphi(x)}{1 + \|y' - f(x)\|}(y' - f(x)) & \text{if } x \in \bigcup_{y \in \tilde{Y} \setminus \{y'\}} A_y \\ f(x) + \dfrac{\varepsilon\varphi(x)}{1 + \|y'' - f(x)\|}(y'' - f(x)) & \text{if } x \in A_{y'} \\ f(x) & \text{if } x \in X \setminus \bigcup_{y \in \tilde{Y}} A_y . \end{cases}$$

By means of the same reasonings used in the proof of Theorem 2.1 of [?], it is possible to check that the function f_ε satisfies the thesis.

Remark 2. Theorem 2 gives a positive answer to Problem 3.1 of [2].

Another consequence of Theorem 1 is the following

THEOREM 3. *Let* X *be a perfect metric space. Then, there exists a partition* $\mathscr{F}$ *of* X *having at most the continuum power and composed of closed and rare sets.*

Proof. By Theorem 1, there exists a continuous function $f : X \to [0, 1]$ such that int $(f^{-1}(t)) = \varnothing$ for every $t \in [0, 1]$. It suffices to take $\mathscr{F} = \{f^{-1}(t)\}_{t \in f(X)}$.

Remark 3. It is interesting to compare Theorem 3 with the classical Baire category theorem.

We have, furthermore, the following result.

THEOREM 4. *Let* X *be a locally connected, perfect and complete metric space. Then, there is a partition* $\mathscr{F}$ *of* X, *having the continuum power and composed of closed and rare sets, and a relatively compact* $\mathscr{G}_\delta$*-set* $X^* \subseteq X$ *such that, for every* $Q \in \mathscr{F}$, *the set* $X^* \cap Q$ *is a singleton.*

Proof. Let $\mathscr{E}$ be the family of all connected components of X. By Theorem 1, for each $\Gamma \in \mathscr{E}$, there exists a continuous function f_Γ from Γ onto [0,1] such that int $(f_\Gamma^{-1}(t)) = \varnothing$ for every $t \in [0, 1]$. For every $t \in [0, 1]$, put $Q(t) = \bigcup_{\Gamma \in \mathscr{E}} f_\Gamma^{-1}(t)$. It is easy to check that the set $Q(t)$ is closed and rare. Now, fix $\Gamma^* \in \mathscr{E}$. By Theorem 3.4 of [3], there exists a relatively compact $\mathscr{G}_\delta$-set $X^* \subseteq \Gamma^*$, such that, for every $t \in [0, 1]$, the set $X^* \cap f_{\Gamma^*}^{-1}(t)$ is a singleton. Plainly, the family $\mathscr{F} = \{Q(t)\}_{t \in 0,1}$ and the set X^* have the desired properties.

Before stating the final result we recall the notion of inductive openness for a function. Let Σ be a topological space. A function $f : \Sigma \to \mathbf{R}$ si said to be *inductively open* if there exists $\Sigma^* \subseteq \Sigma$ such that $f(\Sigma^*) = f(\Sigma)$ and the function $f_{|\Sigma^*} : \Sigma^* \to f(\Sigma)$ is open.

THEOREM 5. *Let* X *be a connected and locally connected metric space. Then any continuous real function on* X *is the uniform limit of a sequence of continuous and inductively open real functions on* X.

Proof. Apply jointly Theorem 2 and Theorem 2.4 of [4].

Observe that Theorem 5 specifies Theorem 3.1 of [2].

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