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On analytic rapidly decreasing functions of a real variable

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RENDICONTI

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Classe di Scienze fisiche, matematiche e naturali

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Presiede il Presidente della Classe GIUSEPPE MONTALENTI

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Analisi matematica. — *On analytic rapidly decreasing functions of a real variable.* Nota (*) del Socio GIANFRANCO CIMMINO.

RIASSUNTO. — Condizione necessaria e sufficiente affinché una funzione rapidamente decrescente di variabile reale sia uniformemente analitica è che per i suoi coefficienti $\gamma_0, \gamma_1, \dots$ di Fourier-Hermite riesca $\gamma_m = 0\,(e^{-\sqrt{mt}})$, per $t > 0$ abbastanza piccolo.

1. Among the rapidly decreasing functions from **R** into **C** it may be of some interest, as I have recently pointed out in [1], [2], to consider those which are in particular analytic. Such is not, for instance, each non-zero $\varphi(s) \in C_0^\infty(\mathbf{R})$, whereas such are, on the contrary, the Hermite's functions $\varphi_m(s)$, $m \in \mathbf{N}$ defined by

$$(1.1)\quad \varphi_m(s) = \frac{1}{\sqrt{m!\,2^m}} e^{-s^2/2} H_m(s)\ ,\ H_0(s) = 1\ ,\ H_m(s) = e^{s^2} \frac{d^m e^{-s^2}}{ds^m}\ ,\ m = 1, 2, \cdots$$

and more generally all products of arbitrary polynomials by exponential functions with a real negative polynomial as exponent.

The analyticity condition for $\varphi(s) \in C^\infty(\mathbf{R})$ can be expressed by saying that for all $s \in \mathbf{R}$

$$(1.2)\qquad \exists M > 0\ ,\ \text{s.t.}\ |\varphi^{(k)}(s)| \leq k!\,M^k\ ,\qquad \forall k = 1, 2, \cdots,$$

(*) Presentata nella seduta del 26 novembre 1983.

where M depends in general on s. When furthermore condition (1.2) is verified with a constant M, so that for the convergence radius $r(s)$ of the Taylor expansion

$$\varphi(s+z)=\sum_{c}^{\infty}{}_k\,\varphi^{(k)}(s)\,\frac{z^k}{k!} \tag{1.3}$$

one will have $\inf_{s\in\mathbf{R}} r(s) \geq \frac{1}{M} > 0$, then we will call $\varphi(s)$ a uniformly analytic function of the real variable s.

In the case that condition (1.2) is verified not only for sufficiently large values of M, but for an arbitrary $M > 0$, we will call $\varphi(s)$ an entire function of the real variable s.

Thus among the functions $\varphi(s)\in C^{\infty}(\mathbf{R})$ the analytic ones, the uniformly analytic ones, the entire ones are restrictions to the real axis s of functions of the complex variable $s+it$, which are analytic respectively in an arbitrary open set containing $t=0$, in an open set containing a strip $|t|<\delta$, in the whole plane $s+it$.

As usual, we call $\mathscr{S}$ the linear space of the rapidly decreasing functions $\varphi(s)$ equipped with the well-known L. Schwartz topology.

2. *Let $\varphi(s)$ be a function $\in\mathscr{S}$, for which the sequence of the Fourier coefficients with respect to the orthogonal system* (1.1).

$$\gamma_m=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\varphi(s)\,\varphi_m(s)\,ds\,,\qquad m\in\mathbf{N} \tag{2.1}$$

satisfies the asymptotic condition

$$\exists t>0\quad,\quad \text{s.t.}\quad \gamma_m=0\left(e^{-\sqrt{m}\,t}\right). \tag{2.2}$$

Then for every $h\in\mathbf{N}$ the product $s^h\varphi(s)$ will be uniformly analytic.

For $h=0$ this statement was proved in [2] to be a consequence of the known formulas

$$\varphi'_m=\sqrt{\frac{m}{2}}\,\varphi_{m-1}-\sqrt{\frac{m+1}{2}}\,\varphi_{m+1}\quad,\quad s\varphi_m=\sqrt{\frac{m}{2}}\,\varphi_{m-1}+\sqrt{\frac{m+1}{2}}\,\varphi_{m+1}\,, \tag{2.3}$$

a fact that a few months later has independently been observed also by A. Avantaggiati [3] in the frame of a series of more general results of his on the subject.

We now extend the proof given in [2] to the case of an arbitrary $h \in \mathbf{N}$.

To this we start by remarking that, if one indicates by L an upper bound of $|\varphi_m(s)|$, $m \in \mathbf{N}$, $s \in \mathbf{R}$, one gets the majorization formula

$$(2.4) \qquad |(s^h \varphi_m)^{(k)}| \leq \sqrt{2^{h+k}(m+1)(m+2)\cdots(m+h+k)}\, \mathrm{L}\,, \qquad \forall m, h, k, \in \mathbf{N} \qquad \forall s \in \mathbf{R}\,,$$

as it can be seen by induction, because from (2.3) one draws that the inequality (2.4) holds for $h+k=1$, $\forall m \in \mathbf{N}$, while it also does so for $h+k=n+1$ every time it does for $h+k=n$.

Now by the hypothesis (2.2) we have that for a suitable $\mathrm{L}' > 0$,

$$(2.5) \qquad |\gamma_m| \leq \mathrm{L}' e^{-\sqrt{m}t} \quad , \quad \forall m \in \mathbf{N}\,,$$

and from (2.4), (2.5) it follows

$$2.6) \qquad |(s^h \varphi)^{(k)}| = \left| \sum_0^\infty{}_m \gamma_m (s^h \varphi_m)^{(k)} \right| \leq \mathrm{LL}' \sqrt{2^{h+k}} \sum_0^\infty{}_m (\sqrt{m}+1)\cdots \cdots(\sqrt{m}+h+k)\, e^{-\sqrt{m}t}\,,$$

whence by a device indicated in [2] one obtains

$$(2.7) \qquad |(s^h \varphi)^{(k)}| \leq \mathrm{LL}' \sqrt{2^{h+k}} \frac{d^{h+k}}{d\rho^{h+k}} \frac{\rho^{h+k}}{(1-\rho)^4}\,, \quad \rho = e^{-t/2}\,.$$

From this, noting further that

$$(2.8) \qquad \frac{d^n}{d\rho^n} \frac{\rho^n}{(1-\rho)^4} = \frac{n\,!}{(1-\rho)^{n+4}} p_n(\rho)\,, \quad p_n(\rho) = \binom{n+3}{3} - 3\binom{n+2}{3}(1-\rho) + 3\binom{n+1}{3}(1-\rho)^2 - \binom{n}{3}(1-\rho)^3$$

and defining M_h by

$$(2.9) \qquad \mathrm{M}_h = \frac{\sqrt{2}}{1-\rho} \sup_{k=1,2,\cdots} \left[\mathrm{LL}' \sqrt{2^h} \frac{(k+1)\cdots(k+h)}{(1-\rho)^{h+4}} p_{h+k}(\rho) \right]^{1/k},$$

we conclude, as desired, that

$$(2.10) \qquad |(s^h \varphi)^{(k)}| \leq k\,!\, \mathrm{M}_h^k \quad , \quad \forall k = 1, 2, \cdots,$$

where the finiteness of M_h in (2.9) is clear, since $\lim_{k\to\infty} |\text{polynomial in } k|^{1/k} = 1$.

3. *The Fourier-Hermite coefficients* (2.1) *of every uniformly analytic function* $\varphi(s) \in \mathscr{S}$ *do satisfy condition* (2.2).

To prove this statement we firstly remark that, as a consequence of (1.2), the Cauchy problem

$$u_{ss} + u_{tt} - s^2 u = 0 \quad , \quad u(s, 0) = \varphi(s) \quad , \quad u_t(s, 0) = 0 , \tag{3.1}$$

shall have a unique solution $u(s, t)$, $s \in \mathbf{R}$, $|t| < \dfrac{1}{\mathrm{M}}$ in the analytic field, and for each $s_0 \in \mathbf{R}$ the coefficients of the corresponding Taylor expansion

$$u(s, t) = \sum_0^\infty{}_{hk} c_{hk} (s - s_0)^k t^{2h} \tag{3.2}$$

cannot be defined otherwise than by

$$c_{0k} = \frac{\varphi^{(k)}(s_0)}{k!} , \quad c_{h+1,k} = \frac{1}{(2h+1)(2h+2)} (c_{h,k-2} + \tag{3.3}$$

$$+ 2 s_0 c_{h,k-1} + s_0^2 c_{hk} - (k+1)(k+2) c_{h,k+2}) .$$

In (1.2) we may obviously suppose that $\mathrm{M} \geq 1$. Then from (3.3) it readily follows by induction

$$|c_{hk}| \leq \binom{h + 2h + |s_0| + 1}{2h} \mathrm{M}^{k+2h} \tag{3.4}$$

and consequently, for every t with $|t| < \dfrac{1}{\mathrm{M}}$,

$$\sum_c^\infty{}_h |c_{hk}| t^{2h} \leq \mathrm{M}^k \sum_c^\infty{}_j \binom{k + j + |s_0| + 1}{j} (\mathrm{M}|t|)^j = \tag{3.5}$$

$$= \mathrm{M}^k (1 - |t|)^{-(k+|s_0|+1)} .$$

Thus the double power series (3.2) turns out to be absolutely convergent in every rectangle $\mathrm{T}_{s_0,\delta,\varepsilon}$ of the type

$$\mathrm{T}_{s_0,\delta,\varepsilon} = \Big\{ (s, t) \in \mathbf{R}^2 , \text{ s. t. } \; |t| \leq \delta < \frac{1}{\mathrm{M}} , \quad |s - s_0| \leq \tag{3.6}$$

$$\leq \varepsilon \frac{1 - \delta \mathrm{M}}{\mathrm{M}} , \varepsilon < 1 \Big\} .$$

These rectangles, by fixed δ, ε and variable s_0, cover the strip $|t| \leq \delta$, and since δ is an arbitrary positive number $< \frac{1}{M}$, we may conclude that (3.2), (3.3) afford the analytic solution $u(s, t)$ of (3.1) we were looking for. Moreover, for each $s_0 \in \mathbf{R}$ and $t \in [-\delta, \delta]$, we have also

$$(3.7) \qquad u_{s^k}(s_0, t) = k! \sum_0^\infty {}_h c_{hk} t^{2h}, \qquad k = 0, 1, 2, \ldots$$

with c_{hk} given by (3.3). And the convergence of the series $\sum_0^\infty {}_h c_{hk} \delta^{2h}$, $k = 0, 1, \cdots$ ensures that

$$3.8) \qquad \forall k \in \mathbf{N}, \quad \exists L_k > 0, \quad \text{s. t.} \quad |c_{hk}| \leq L_k / \delta^{2h}, \qquad \forall h \in \mathbf{N}.$$

Now, one can easily check that the two double power series

$$(3.9) \qquad \sum_0^\infty {}_{hk} \frac{(s - s_0)^k t^{2h}}{k!(2h)!} \frac{d^k}{ds_0^k} \left[\left(s_0^2 - \frac{d^2}{ds_0^2}\right)^h \varphi(s_0)\right]$$

$$\sum_0^\infty {}_{hk} \frac{(s - s_0)^k t^{2h}}{k!(2h)!} \sum_0^\infty {}_m (2m + 1)^h \gamma_m \varphi_m^{(k)}(s_0)$$

with γ_m defined by (2.1), are formally solution of (3.1). Therefore these coefficients ought to be the same c_{hk}'s defined by (3.3), so that from (3.7), (3.8) we (deduce (writing s instead of s_0)

$$(3.10) \qquad \left|\frac{d^k}{ds^k}\left[\left(s^2 - \frac{d^2}{ds^2}\right)^h \varphi(s)\right]\right| \leq \frac{k!(2h)! L_k}{\delta^{2h}}. \qquad \forall h, k \in \mathbf{N},$$

$$(3.11) \qquad |t| < \delta \Rightarrow u_{s^k}(s, t) = \sum_0^\infty {}_m \gamma_m \varphi_m^{(k)}(s) \cosh(\sqrt{2m + (1)}\, t),$$

$$|u_{s^k}(s, t)| \leq k! L_k / \left(1 - \frac{t^2}{\delta^2}\right).$$

And now the convergence of the series (3.11), were it verified even but for $k = 0, 1$, $s = 0$, entails clearly (2.2) since one has

$$(3.12) \qquad \varphi_{2p}(0) = \frac{(-1)^p \sqrt{(2p)!}}{p! 2^p} \simeq \frac{1}{(\pi p)^{1/4}}, \qquad \varphi'_{2p+1}(0) =$$

$$= 2\sqrt{p + \tfrac{1}{2}}\, \varphi_{2p}(0) \simeq 2\left(\frac{p}{\pi}\right)^{1/4},$$

by which the proof is completed.

4. The characterization given in 2., 3. of all rapidly decreasing functions which are uniformly analytic shows that the linear space formed by these functions is actually larger than the space named $\mathcal{S}_a$ in [1], 12 and it turns out to be precisely the space for which in [2], 3 we used the notation $\mathcal{S}^{\frac{1}{2}}_{a}$.

From this we see the way the conjecture suggested in [1], 12 has to be corrected, all the rest remaining unchanged. Thus, in particular, we recognize the main result to hold, that the dual spaces of $\mathcal{S}_a$, $\mathcal{S}^{\frac{1}{2}}_{a}$ are made up by the generalized functions, that present themselves as traces on the s-axis of all solutions $u(s, t)$ respectively [1] of $u_{ss} - u_t - s^2 u = 0$ in the half plane $t > 0$ and [2] of $u_{ss} + u_{bt} - s^2 u = 0$ in either $t > 0$ or $t < 0$.

For the Fourier–Hermite coefficients of entire functions $\in \mathcal{S}$ we have a characterization similar to (2.2), namely

$$\gamma_m = 0\,(e^{-\sqrt{m}\,t},) \qquad \forall t > 0 . \tag{4.1}$$

It remains an open question to find out an analogous condition characterizing the sequences of Fourier–Hermite coefficients of all functions which are but simply, even if not uniformly, analytic in the sense defined in 1.

References

[1] G. Cimmino (1982) – *On tempered ultradistributions as boundary traces*, « Proceeding, of the international Meeting dedicated to the memory of professor Carlo Miranda », Naples.

[2] G. Cimmino (1982) – *On linear operators between spaces of tempered distribution or ultradistributions*, « Rendiconti del Sem. Mat. e Fis. di Milano », on print.

[3] A. Avantaggiati (1983) – *Sviluppi in serie di Hermite-Fourier e condizioni di analiticità e quasi analiticità*, « Proceedings » mentioned above, p. 312.