Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

# Rendiconti

Umberto Marconi

## On uniform paracompactness of the $\omega_\mu$-metric spaces

*Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8*, Vol. **75** (1983), n.3-4, p. 102–105.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1983_8_75_3-4_102_0>

**Topologia.** — *On uniform paracompactness of the $\omega_\mu$–metric spaces.* Nota (*) di UMBERTO MARCONI, presentata dal Socio G. SCORZA DRAGONI.

RIASSUNTO. — Gli spazi $\omega_\mu$-metrici uniformemente numerabilmente paracompatti sono uniformemente paracompatti. Si fornisce altresì una caratterizzazione degli spazi $\omega_\mu$–metrici fini.

## 0. INTRODUCTION

A uniform space X is said to be uniformly paracompact if every directed (= closed under finite union) open covering is uniform; it is said to be uniformly countably paracompact if every directed countable open cover is uniform (see [R], [$H_1$], [M]). In [$H_2$] it is proved that in the realm of metric spaces these two concepts coincide. We will extend this result to $\omega_\mu$-metric spaces. An $\omega_\mu$-metric space is a uniform space which admits a base of uniform coverings

$$\mathcal{B} = \{\mathcal{U}_\alpha : \alpha < \omega_\mu\}$$

well ordered (by star-refinement) by a regular cardinal $\omega_\mu$; if $\mu > 0$ it is easy to prove that coverings $\mathcal{U}_\alpha$ may be assumed to be clopen partitions.

In section 1 it is proved that if $\mu > 0$, $\omega_\mu$-metric spaces uniformly countably paracompact are uniformly paracompact. In section 2 fine $\omega_\mu$-metric spaces are characterized.

## 1. RESULTS

In this section only $\omega_\mu$-metric spaces with $\mu > 0$ are considered. In this case a point with a compact neighbourhood is an isolated point.

PROPOSITION 1.1. *Let* X *be an* $\omega_\mu$*-metric space topologically discrete* (*=locally compact*). *If* X *is uniformly countably paracompact, then* X *is uniformly locally compact, that is there exists a uniform covering made by finite* (*compact*) *sets.*

(*) Pervenuta all'Accademia il 19 ottobre 1983.

*Proof.* If X isn't uniformly locally compact, for every $\alpha < \omega_\mu$ we can define by induction a countable infinite subset $N_\alpha$ of X and an ordinal $f(\alpha)$ such that:

1) $\alpha \leq f(\alpha) < \omega_\mu$;
2) $\alpha \neq \beta \Rightarrow N_\alpha \cap N_\beta = \emptyset$
3) there exists an element $U_\alpha \in \mathcal{U}_{f(\alpha)}$ such that $N_\alpha \subset U_\alpha$.

Let $f(0) = 0$, $U_0$ an infinite element of $\mathcal{U}_0$ and $N_0$ a countable infinite subset of $U_0$.

Let $\alpha < \omega_\mu$. If $f(\beta)$ and $N_\beta$ are defined for every $\beta < \alpha$, put $Y_\alpha = \bigcup_{\beta<\alpha} N_\beta$. Since $|Y_\alpha| < \omega_\mu$ and X is discrete, the covering $\{X \setminus Y, \{y\} : y \in Y_\alpha\}$ is uniform; i.e. it has a refinement $\mathcal{U}_{f(\alpha)} \in \mathcal{B}$, with $f(\alpha) \geq \alpha$. Let $U_\alpha$ be an infinite element of $\mathcal{U}_{f(\alpha)}$ and $N_\alpha$ a countable infinite subset of $U_{f(\alpha)}$.

Put $N_\alpha = \{x_n^\alpha : n \in \omega\}$.

For every $n \in \omega$, let $C_n = \{x_k^\alpha : k \geq n, \alpha < \omega_\mu\}$.

The directed countable open covering of X, $\{X \setminus C_n : n \in \omega\}$, cannot be uniform, contradicting the hypothesis.

PROPOSITION 1.2. *Let $X^d$ be the subset of all accumulation points of an $\omega_\mu$-metric space* X. *If* X *is uniformly countably paracompact, then $X^d$ is $\omega_\mu$-compact.*

*Proof.* $X^d$, being a closed subspace of X, is uniformly countably paracompact. Therefore $X^d$ can be supposed to coincide with X.

An $\omega_\mu$-metric space is ultraparacompact (see [AM]). Therefore, if X isn't $\omega_\mu$-compact, there exists a partition of clopen sets $\mathcal{P} = \{A_\beta : \beta < \omega_\mu\}$ of cardinal $\omega_\mu$. We can choose an increasing mapping $\alpha : \omega_\mu \to \omega_\mu$ such that for every $\beta < \omega_\mu$ there exists an element $U_{\alpha(\beta)} \in \mathcal{U}_{\alpha(\beta)}$ such that $U_{\alpha(\beta)} \subset A_\beta$.

Obviously $U_{\alpha(\beta)}$ contains a countably infinite subset $N_\beta = \{x_n^\beta : n \in \omega\}$.

Each $N_\beta$ is closed and discrete, $N_\beta \subset A_\beta$, and $\{A_\beta : \beta < \omega_\mu\}$ is a clopen partition; it follows that $N = \bigcup_{\beta<\omega_\mu} N_\beta$ is closed and discrete. But $U_{\alpha(\beta)} \cap N = N_\beta$ is infinite for each $\beta < \omega_\mu$, contradicting Prop. 1.1.

Now we can prove the main result.

THEOREM 1.1. *Let* X *be an $\omega_\mu$-metric space. The following conditions are equivalent:*

1) X *is uniformly countably paracompact;*

2) *there exists an $\omega_\mu$-compact subset* K *of* X *such that for every $\alpha < \omega_\mu$ the subset $X \setminus St(K, \mathcal{U}_\alpha)$ is uniformly locally compact;*

3) X *is uniformly paracompact.*

*Proof.* $1 \Rightarrow 2$. Let $K = X^d$ be the derived set of X. By Proposition 1.2, K is $\omega_\mu$-compact. For every $\alpha < \omega_\mu$ the closed subspace $X \setminus St(K, \mathcal{U}_\alpha)$, being topologically discrete and uniformly countably paracompact, is uniformly locally compact (Prop. 1.1).

$2 \Rightarrow 3$. Let $\mathscr{A} = \{A_\gamma : \gamma \in \Gamma\}$ be a directed open covering. For every $x \in K$, there exists a covering $\mathscr{U}_{\alpha(x)} \in \mathscr{B}$ such that $\mathrm{St}(x, \mathscr{U}_{\alpha(x)}) \subset A_\gamma$ for some $\gamma \in \Gamma$. The covering of K, $\{\mathrm{St}(x, \mathscr{U}_{\alpha(x)}) : x \in K\}$ has a subcovering of cardinal less than $\omega_\mu$, say $\mathscr{V} = \{\mathrm{St}(y, \mathscr{U}_{\alpha(y)}) : y \in Y$, with $Y \subset K$, $|Y| < \omega_\mu\}$.

Let $\alpha = \sup\{\alpha(y) : y \in Y\}$. If $U \in \mathscr{U}_\alpha$ and $U \cap K \neq \varnothing$, there exists a point $y \in Y$ such that $U \subset \mathrm{St}(y, \mathscr{U}_{\alpha(y)})$, because $\mathscr{U}_\alpha$ is a refinement of all partitions $\mathscr{U}_{\alpha(y)}$. Therefore the family $\{U \in \mathscr{U}_\alpha : U \cap K \neq \varnothing\}$ is a refinement of the family $\{A_\gamma \in \mathscr{A} : A_\gamma \cap K \neq \varnothing\}$.

Consider $X \setminus \mathrm{St}(K, \mathscr{U}_\alpha)$. By hypothesis, there exists an index $\beta$, $\alpha < \beta < \omega_\mu$, such that the set $\mathrm{St}(p, \mathscr{U}_\beta) \cap (X \setminus \mathrm{St}(K, \mathscr{U}_\alpha)) = \mathrm{St}(p, \mathscr{U}_\beta)$ is finite (last equality holds because $\mathscr{U}_\beta$ is a refinement of the partition $\mathscr{U}_\alpha$).

It is easy to conclude that $\mathscr{U}_\beta$ is the required uniform refinement of $\mathscr{A}$.

$3 \Rightarrow 1$. Obvious.

## 2. Fine $\omega_\mu$-metric spaces

A uniform space X topologically paracompact is fine if every open covering is uniform. It is easy to prove that an $\omega_\mu$-compact $\omega_\mu$-metric space is fine ([R] Th. 5.2). Since a paracompact fine uniform space is uniformly paracompact, Proposition 1.2 implies the following:

PROPOSITION 2.1. *Let* X *be an* $\omega_\mu$*-metric space without isolated points. The following conditions are equivalent*:

1) X *is* $\omega_\mu$*-compact*;

2) X *is fine*.

It is easy to prove that Proposition 2.1 holds also for metric spaces ($\omega_\mu = \omega$). In fact, if a metric space (X, d) isn't compact, there exists a closed infinite discrete subset $\{x_n : n \in \omega\}$. If the points $x_n$ are accumulation points, for every $n \in \omega$ there exists a point $y_n \in X$ with $0 < d(x_n, y_n) < \dfrac{1}{n}$. Then the discrete subset $\{x_n : n \in \omega\} \cup \{y_n : n \in \omega\}$ cannot be uniformly discrete, and therefore the space (X, d) cannot be fine.

The following theorem provides a characterization of fine $\omega_\mu$-metric spaces (here $\omega_\mu$ can be equal to $\omega$).

THEOREM 2.1. *Let* X *be an* $\omega_\mu$*-metric space. The following conditions are equivalent :*

1) X *is fine*;

2) *There exists an* $\omega_\mu$*-compact subset* K *of* X *such that for every* $\alpha < \omega_\mu$ *the subspace* $X \setminus \mathrm{St}(K, \mathscr{U}_\alpha)$ *is uniformly discrete.*

*Proof.* $1 \Rightarrow 2$. Put $K = X^d$. By Proposition 2.1, K is $\omega_\mu$-compact. Let $\alpha < \omega_\mu$.

Since the discrete subspace $X \setminus St(K, \mathcal{U}_\alpha)$ is closed, it is fine, thus uniformly discrete.

$2 \Rightarrow 1$. Let $\mathcal{A}$ be an open covering of X. By $\omega_\mu$-compactness of K, there exists a covering $\mathcal{U}_\alpha \in \mathcal{B}$ such that for every $x \in K$ the $St(x, \mathcal{U}_\alpha)$ is contained in some element of $\mathcal{A}$. Since $X \setminus St(K, \mathcal{U}_{\alpha+2})$ is uniformly discrete, there exists a covering $\mathcal{U}_\beta \in \mathcal{B}$ with $\alpha + 2 \leq \beta$, such that $St(p, \mathcal{U}_\beta) \cap X \setminus St(K, \mathcal{U}_{\alpha+2}) = \{p\}$ for every $p \notin St(K, \mathcal{U}_{\alpha+2})$. Then for every $p \notin St(K, \mathcal{U}_{\alpha+1})$ we have $\{p\} = St(p, \mathcal{U}_\beta)$. Therefore the open cover of $X, \mathcal{V} = \{St(x, \mathcal{U}_\alpha): x \in K\} \cup \{St(p, \mathcal{U}_\beta): p \notin St(K, \mathcal{U}_{\alpha+1})\}$, is a uniform refinement of $\mathcal{A}$, since it it is refined by $\mathcal{U}_\beta$.

## REFERENCES


[AM] ARTICO G. and MORESCO R. (1982) – *$\omega_\mu$-additive topological spaces*, « Rend. Sem. Mat. Univ. Padova », *67*, 131-141.

[$H_1$] HOHTI A. (1981) – *On uniform paracompactness*, « Ann. Acad. Sc. Fenn. », ser A, « Math. Diss. », *36*.

[HI] HOHTI A. (1981) – *A theorem on uniform paracompactness*, Gen. Top. and its relations to ..., 5, Proc. Fifth Prague Topological Symp., 384-386.

[I] ISBELL J.R. (1964) – *Uniform spaces*, « Math. Surveys », *12*, « Amer. Math. Soc. », Providence, Rhod Island. 1964.

[R] REICHEL H.C. (1978) – *Some results on uniform spaces with linearly ordered bases*, « Fund. Math. », *98*, 25-39.