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Uniform algebras and analytic multifunctions

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Analisi matematica. — *Uniform algebras and analytic multifunctions.* Nota (*) di ZBIGNIEW SLODKOWSKI, presentata dal Corrisp. E. VESENTINI.

RIASSUNTO. — Dati due elementi f e g in un'algebra uniforme A, sia $G = f(M_A)/f(\partial_A)$. Nella presente Nota si dànno, fra l'altro, due nuove dimostrazioni elementari del fatto che la funzione $\lambda \to \log \max g(f^{-1}(\lambda))$ è subarmonica su G e che l'applicazione $\lambda \to g(f^{-1}(\lambda))$ è analitica nel senso di Oka.

INTRODUCTION

An analytic multifunction (alternatively: -set-valued or multivalued analytic function) is an upper semi-continuous, compact valued correspondence $\lambda \to K_\lambda : G \to 2^{\mathbf{C}}$ (where G is open in $\mathbf{C}$) such that the set $U = \{(\lambda, z) : \lambda \in G, z \notin K_\lambda\}$ is pseudoconvex.

This notion was introduced by K. Oka [5] and studied further by T. Nishino [4] and H. Yamaguchi [11]. The author, motivated by some problems posed by B. Aupetit [1] reintroduced the notion in [6] and studied it further in [8, 9, 10]. Interesting related results were obtained by J. Wermer [14], B. Aupetit [2], Aupetit and Zemanek [3] and E. Vesentini [12].

The present Note is concerned with the following results, obtained earlier by the author.

THEOREM 1 ([6, 8, 9]). *Let $\lambda \to T_\lambda : G \to B(X)$ be an analytic operator-valued function (where $G \subset \mathbf{C}$ is open and X is a complex Banach space). Then $\lambda \to \sigma(T_\lambda) : G \to 2^{\mathbf{C}}$ is an analytic multifunction.*

THEOREM 2 ([6, 8]). *Let* A *be a uniform algebra and f, $g \in$ A. Then*

$$\lambda \to g(f^{-1}(\lambda)) : f(M_A)/f(\partial_A) \to 2^{\mathbf{C}}.$$

is an analytic multifunction.

THEOREM 3 ([6, 8a, 10]). *Let $\lambda \to K_\lambda : G \to 2^{\mathbf{C}}$ be an analytic multifunction and let $\psi(\lambda, z_1, \ldots, z_n)$ be a plurisubharmonic function defined in some neigh-*

(*) Pervenuta all'Accademia il 27 giugno 1983.

bourhood of the set $\{(\lambda, z_1, \ldots, z_n) : \lambda \in G, \ z_1, \ldots, z_n \in K_\lambda\}$. *Then the function*

$$\varphi(\lambda) = \max \{\psi(\lambda, z_1, \ldots, z_n) : z_1, \ldots, z_n \in K_\lambda\}$$

is subharmonic in G.

Theorems 1 and 2 are in this respect important, that they allow direct application of complex analysis to problems concerning spectra and uniform algebras. In this paper we give new, essentially simpler proofs of these Theorems (Sect. 2 and 3), and discuss some applications of Theorem 3 in connection with the recent work of E. Vesentini [12] (Sect. 4). We also include a simple proof of the special case of Rossi's local maximum modulus principle (Sect. 1), and indicate, how it yields another proof of Theorem 2.

1. Special case of Rossi's local maximum modulus principle

Wermer's Theorem [13]. *Let* A *be a uniform algebra and* $f, g \in A$. *Then the function*

$$\varphi(\lambda) = \log \max |g(f^{-1}(\lambda))|$$

is subharmonic in $G = f(M_A) / f(\partial_A)$.

Wermer's original proof is based on Rossi's local maximum modulus principle. Below we present an elementary proof of Wermer's theorem, which, at the same time, yields the following, rather special, case of Rossi's local maximum modulus principle.

Corollary. *Let* A *be a uniform algebra and* $f \in A$. *If* A *is a compact subset of* $f(M_A) / f(\partial_A)$ *then for every* $g \in A$

$$\max |g| \,|_{f^{-1}(F)} = \max |g| \,|_{f^{-1}(\partial F)} . \tag{1}$$

Basically, our proof is contained in the proof of more general Theorem 4 in [7]. Since the special case is simpler and of independent interest, it seems worthwhile to present it separately. We base our approach on the following Proposition and Lemma 1.

Proposition. *Under the assumptions of Wermer's theorem the function*

$$\psi(\lambda) = \log \inf \| g + (f - \lambda) A \|$$

satisfies the following inequality

$$\psi(\lambda_0) \leq -\log(1 - r/R) + \frac{1}{2\pi}\int \psi(\lambda_0 + re^{i\theta})\,d\theta\,, \tag{2}$$

for every $\lambda_0 \in G$, *and every* $0 < r < R = \operatorname{dist}(\lambda_0, \partial G)$.

LEMMA 1. *Let* A, f, g, G *be as in Wermer's theorem. Then for every* $m_0 \in A$ *annihilating* $(f - \lambda_0)A$, *there is an analytic vector-valued function* $\lambda \to m_\lambda\colon G \to A^*$, *such that*

(i) $m_{\lambda_0} = m_0$

(ii) $m_\lambda \mid (f - \lambda) A_\lambda \equiv 0$, *for* every $\lambda \in G$,

(iii) *if* $R = \operatorname{dist}(\lambda_0, \partial G)$, *then for every* $\lambda \in D(\lambda_0, R)$,

$$\| m_\lambda \| \leq (1 - |\lambda - \lambda_0|/R)^{-1} \| m_0 \|.$$

A weaker form of Lemma 1 was given in [7]: the function m_λ was constructed only in the disc $D(\lambda_0, R)$, which suffices for the proof of the Proposition, but not for further applications in Sect. 2; estimate (iii) was not explicitly formulated, but it follows immediately from the proof in [7]. Here we reproduce a very simple construction of function m_λ in the whole G, communicated to the author by J. Wermer.

Proof of Lemma 1 (J. Wermer). Choose a (complex-valued) measure μ on ∂_A representing m_0 and such that $\| m_0 \| = \operatorname{var}(\mu)$. Define m_λ by the formula

$$m_\lambda(h) = \int_{\partial_A} \frac{f - \lambda_0}{f - \lambda}\, h\,d\mu\,,\ h \in A\,,\ \lambda \in G\,.$$

It is clear that the function $\lambda \to m_\lambda : G \to A^*$ is holomorphic, and that the conditions (i) and (ii) are satisfied. Concerning condition (iii) observe that for every $x \in \partial_A$, $|f(x) - \lambda_0| \geq R$ and so

$$\frac{|f(x) - \lambda_0|}{|f(x) - \lambda|} = \frac{|f(x) - \lambda_0|}{|(f(x) - \lambda_0) - (\lambda - \lambda_0)|} = \left|1 - \frac{\lambda - \lambda_0}{f(x) - \lambda_0}\right|^{-1} \leq$$
$$\leq \left(1 - \frac{|\lambda - \lambda_0|}{R}\right)^{-1},$$

provided $\lambda \in D(\lambda_0, R)$. Therefore

$$\| m_\lambda \| \leq (1 - |\lambda - \lambda_0|/R)^{-1} \operatorname{var}(\mu) = (1 - |\lambda - \lambda_0|/R)^{-1} \| m_0 \|,$$

as desired.

Proof of the Proposition. Fix $\lambda_0 \in G$. Since $\inf \| g + (f - \lambda_0) A \|$ is the norm of the coset $[g]$ in the quotient space $A/\overline{(f - \lambda_0)}$, there is a linear functional m vanishing on $(f - \lambda_0) A$ such that

$$\| m_0 \| = 1 \quad \text{and} \quad | \langle m_0 , g \rangle | = \inf \| g + (f - \lambda_0) A \| = \exp \psi (\lambda_0) . \tag{3}$$

Choose an analytic function $\lambda \rightarrow m_\lambda : G \rightarrow A^*$, with $m_{\lambda_0} = m_0$, satisfying conditions of Lemma 1. By condition (ii)

$$| \langle m_\lambda , g \rangle | = | \langle m_\lambda , g + (f - \lambda) h \rangle | \leq \| m_\lambda \| \cdot \| g + (f - \lambda) h \|,$$

for every $h \in A$, and so

$$\log | \langle m_\lambda , g \rangle | \leq \log \| m_\lambda \| + \log \inf \| g + (f - \lambda) A \| = \log \| m_\lambda \| + \psi (\lambda).$$

Taking into account condition (iii) we obtain

$$\log \| \langle m_\lambda , g \rangle \| \leq \log (1 - r/R)^{-1} + \psi (\lambda) , \text{ for } | \lambda - \lambda_0 | = r < R . \tag{4}$$

Finally, since $\lambda \rightarrow \langle m_\lambda , g \rangle$ is analytic, it holds

$$\psi (\lambda_0) = \log | \langle m_0 , g \rangle | \qquad \text{(by (3))}$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} \log | \langle m_{\lambda_0 + re^{i\theta}} , g \rangle | \, d\theta \leq - \log (1 - r/R) +$$

$$+ \frac{1}{2\pi} \int_0^{2\pi} \psi (\lambda_0 + re^{i\theta}) \, d\theta ,$$

for $0 < r < R$ by (4). Q.E.D.

Proof of Wermer's theorem. Put

$$\psi_n (\lambda) = \frac{1}{n} \log \inf \| g^n + (f - \lambda) A \| , \qquad \lambda \in G .$$

By Proposition and (2), ψ_n satisfies the following inequality

$$\psi_n (\lambda) \leq - \frac{1}{n} \log (1 - r/R) + \frac{1}{2\pi} \int_0^{2\pi} \psi_n (\lambda + re^{i\theta}) \, d\theta , \tag{5}$$

for every $\lambda \in G$ and $0 < r < R = \operatorname{dist}(\lambda, \partial G)$. On the other hand, by Gelfand theorem

$$\lim (\| [g^n] \|_\lambda)^{1/n} = r_\lambda ([g]) = \sup \{ | \mu ([g]) | : \mu \in M (A/\overline{(f-\lambda) A})\},$$

where $\| \ \|_\lambda$ and r_λ denote respectively the quotient norm and the spectralradius in the algebra $A/\overline{(f-\lambda) A}$. Since the Gelfand space of this algebra is identified with the annihilator of $(f-\lambda) A$ in M_A, which is equal to $f^{-1}(\lambda)$, $\log r_\lambda ([g]) = \varphi (\lambda)$. Thus

$$\lim \psi_n (\lambda) = \lim \frac{1}{n} \log \| [g^n] \|_\lambda = \varphi (\lambda).$$

Therefore by (5)

$$\varphi (\lambda) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi (\lambda + re^{i\theta}) \, d\theta, \quad \text{for every } \lambda \in G \text{ and } 0 < r < R = \operatorname{dist}(\lambda, \partial G)$$

(observe that all functions $\psi_n (\lambda)$ are bounded from the above by constant $\log \| g \|$). Since $\varphi (\lambda)$ is upper semi-continuous, it is subharmonic.

Q.E.D.

2. THE MULTIFUNCTION $\lambda \to g (f^{-1} (\lambda))$

Before we present a simpler proof of Theorem 2, let us point out that the above special case of Rossi's principle is all that is needed in the previous proof, given in [8, Corollary 3.4]. Since it was not made explicit there (cf. Aupetit [2, p. 45, lines 7-8], we sketch anew the relevant part of the argument.

Proof of Theorem 2 (Sketch). By [8, Theorem 3.2 (ii)] the proof is reduced to showing that for every $a, b \in \mathbf{C}$, and $p(.)$, a polynomial, the function

$$\varphi (\lambda) = \max \{ | (z - \lambda a - b)^{-1} \exp p (\lambda) | : z \in g (f^{-1} (\lambda))\}$$

satisfies the condition

$$\varphi (\lambda_0) \leq \max \varphi \mid_{\partial \overline{D}(\lambda_0, r)} \tag{6}$$

for every disc $\overline{D} (\lambda_0, r) \subset G_0 = \{\lambda : a\lambda + b \notin g (f^{-1} (\lambda))\}$. To see this, put $N = f^{-1} (\overline{D} (\lambda_0, r)$ and $B =$ the uniform closure of $A \mid N$ on N. Since N is A-convex, $M_B = N$, and since the function $g - af - b$ does not vanish on $f^{-1} (G_0) \supset N$, it is invertible in B and $(g - af - b)^{-1} \exp p (f)$ defines an element, say h, of B. It is is clear that $\varphi(\lambda) = \max |h(f^{-1}(\lambda))|$. Thus (6) is equivalent to

$$\max |h||_{f^{-1}(\lambda_0)} \leq \max |h||_{f^{-1}(\partial D(\lambda_0, r))} \tag{7}$$

Since inequality (1) is true also for g in the uniform closure of $A|_{f^{-1}(F)}$, formula (7) holds.

Q.E.D.

Our direct proof of Theorem 2 is based on the following lemma.

LEMMA 2. *Let* U *be a domain in* $\mathbf{C}^2$ *and* G *denotes its projection on the first coordinate. Assume that for every point* $(\lambda_0, z_0) \in G \times \mathbf{C}/U$ *there is an analytic function* $b: U \to \mathbf{C}$ *such that* $b(\lambda_0, z) = (z_0 - z)^{-1}$, *whenever* $(\lambda_0, z) \in U$. *Then* U *is a domain of holomorphy.*

Proof. According to a well known criterion, in order to prove that U is is a domain of holomorphy, it is enough to check that for every $u_1 = (\lambda_1, z_1) \in U$ there is an analytic function $h: U \to \mathbf{C}$, such that its restriction $h|_B$, where $B = B(u_1, r)$ is the largest ball contained in U, cannot be extended holomorphically to any ball $B(u_1, r_1)$ with $r_1 > r$.

One of the intersections $\overline{B}(u_1, r) \cap (\mathbf{C}/G) \times \mathbf{C}$ and $\overline{B}(u_1, r) \cap (G \times G/U)$ is non-empty and contains a point (λ_0, z_0). In the first case it is clear that the function

$$h(\lambda, z) = (\lambda_0 - \lambda)^{-1}, \quad (\lambda, z) \in U, \text{ has the required properties}.$$

In the second case let b be a function satisfying assumptions of the lemma, relative to (λ_0, z_0). Since the function $1 - (z_0 - z)\, b(\lambda, z)$ vanishes on $(\{\lambda_0\} \times \mathbf{C}) \cap U$, one can see that the formula

$$a(\lambda, z) = \begin{cases} (\lambda_0 - \lambda)^{-1}(1 - (z_0 - z)\, b(\lambda, z)), & (\lambda, z) \in U, \lambda \neq \lambda_0, \\ (z_0 - z) \dfrac{\partial b}{\partial \lambda}(\lambda_0, z), & (\lambda, z) = (\lambda_0, z) \in U. \end{cases}$$

defines an analytic function in U, and moreover

$$(\lambda_0 - \lambda)\, a(\lambda, z) + (z_0 - z)\, b(\lambda, z) \equiv 1 \quad \text{in } U. \tag{8}$$

Now it is clear that one of the functions $a|B$ and $b|B$ cannot be extended to any larger ball $B((\lambda_0, z_0), r_1)$, $r_1 > r$, for otherwise the extensions would have to fulfill (8) at $(\lambda, z) = (\lambda_0, z_0)$, which is impossible.

Q.E.D.

Elementary proof of Theorem 2. Consider arbitrary $\lambda_0 \in G$ and $z_0 \in g(f^{-1}(\lambda_0))$. By Lemma 2 it suffices to construct an analytic function $b \colon U \to \mathbf{C}$, such that $b(\lambda_0, z) = (z_0 - z)^{-1}$, $(\lambda_0, z) \in U$, where $U = \{(\lambda, z) \colon \lambda \in G, z \notin g(f^{-1}(\lambda))\}$. For this, choose $m_0 \in f^{-1}(\lambda_0) \subset M_A$, such that $m_0(g) = z_0$. By Lemma 2 there is an analytic function $\lambda \to m_\lambda \colon G \to A^*$, such that $m_{\lambda_0} = m_0$ and $m_\lambda \mid (f - \lambda) A \equiv 0$ for $\lambda \in G$. If $(\lambda, z) \in U$, then $z \notin g(f^{-1}(\lambda))$, and since $M_{A/\overline{(f-\lambda)A}} = f^{-1}(\lambda)$, the set $g(f^{-1}(\lambda))$ is equal to the spectrum of $[g]$ in $A/\overline{(f-\lambda)A}$, and therefore $([g] - z[e])^{-1}$ exists in the quotient algebra. Since $m_\lambda \mid (f - \lambda) A \equiv 0$, m_λ induces a linear form $\tilde{m}_\lambda$ on $A/\overline{(f-\lambda)A}$. Thus the formula

$$b(\lambda, z) = \langle \tilde{m}_\lambda, ([g] - z[e])^{-1} \rangle \quad , \quad (\lambda, z) \in U,$$

defines a single-valued function on U. Since $\langle \tilde{m}_0, [g] \rangle = z_0$ and $\tilde{m}_0$ is multiplicative-linear on $A/\overline{(f-\lambda_0)A}$, $b(\lambda_0, z) = (z_0 - z)^{-1}$, whenever $(\lambda_0, z) \in U$. It remains to show that b is analytic.

We will check this locally. Fix $(\lambda_1, z_1) \in U$; then, as we have observed above, $[g] - z_1[e]$ is invertible in $A/\overline{(f-\lambda_1)A}$, that is for some $p, q \in A$ it holds $(g - z_1 e) q = e - (f - \lambda_1 e) p$. Consequently

$$(g - ze) q = d(\lambda, z) - (f - \lambda) p, \tag{9}$$

where $d(\lambda, z) = e - (z - z_1) q - (\lambda - \lambda_1) p$. If $\varepsilon > 0$ is small enough, $d(\lambda, z)$ is invertible for $(\lambda, z) \in D(\lambda_1, \varepsilon) \times D(z_1, \varepsilon)$. By (9), $(g - ze)(q d(\lambda, z)^{-1}) = e - (f - \lambda e)(p d(\lambda, z)^{-1})$, that is

$$([g] - z[e])^{-1} = [q d(\lambda, z)^{-1}], \text{ in } A/\overline{(f-\lambda)A}, \quad \text{for } (\lambda, z) \in D(\lambda_1, \varepsilon) \times D(z_1, \varepsilon).$$

Therefore, in this neighbourhood $b(\lambda, z) = \langle m_\lambda, p d(\lambda, z)^{-1} \rangle$, and so $b(\lambda, z)$ is analytic.

Q.E.D.

Remark. The reader can notice that the above short proof was made possible only by the simple proof of Lemma 1, given by J. Wermer.

3. THE SPECTRUM

The same method based on Lemma 2 yields an even simpler proof of Theorem 1.

Proof of Theorem 1. Fix $\lambda_0 \in G$ and $z_0 \in \sigma(T_{\lambda_0})$. Let A be the smallest closed sub-algebra of B (X) containing T_{λ_0}, identity and $(T_{\lambda_0} - Iz)^{-1}$, $z \notin \sigma(T_{\lambda_0})$. It is clear that A is commutative and that the spectrum of T_{λ_0}, with respect to A is equal to $\sigma(T_{\lambda_0})$ (which is the spectrum with respect to B (X)). Therefore it is a linear and multiplicative functional m: A $\to$ **C** such that $m(T_{\lambda_0}) = z_0$, and so

$$m((T_{\lambda_0} - zI)^{-1}) = (z_0 - z)^{-1}. \tag{10}$$

Choose a continuous linear extension φ: B (X) $\to$ **C** of m and define $b(\lambda, z) = \langle \varphi, (T_\lambda - zI)^{-1} \rangle$, $(\lambda, z) \in U$. The function $b: U \to \mathbf{C}$ is analytic and, by (10), $b(\lambda_0, z) = (z_0 - z)^{-1}$, for $(\lambda_0, z) \in U$. All conditions of Lemma 2 being fulfilled, the set $U = \{(\lambda, z): \lambda \in G, z \notin \sigma(T_\lambda)\}$ is a domain of holomorphy.

Q.E.D.

Remark. An analogous fact is true for an analytic family $\{T_\lambda\}_{\lambda\in G}$ of closed operators, i.e. $\{(\lambda, z): \lambda \in G, z \notin \tilde{\sigma}(T_\lambda)\}$ (where $\tilde{\sigma}$ denotes the extended spectrum is a domain of holomorphy in $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$; consult [9], Theorem 1] for precise formulation, background and proof. The proof just given extends to the unbounded case with the following modification: A is now the smallest closed sub-algebra of B (X) containing identity and (bounded) operators $(T_{\lambda_0} - zI)^{-1}$, $z \notin \tilde{\sigma}(T_{\lambda_0})$. Then $m \in M_A$ still exists such that $m((T_{\lambda_0} - zI)^{-1}) = (z_0 - z)^{-1}$, $z \notin \tilde{\sigma}(T_{\lambda_0})$. We omit the details.

4. Plurisubharmonic functions and analytic multifunctions

In this section we will discuss some applications of Theorem 3. The theorem was announced in [6], and proven in the preprint [8a, Sect. 7] but, incidentally, was not included in [8], although it follows immediately from some results in [8] concerning analytic multifunctions in several dimensions.

Let us recall that an upper semi-continuous multifunction (compact-valued) $\lambda \to K_\lambda: G \to 2^{\mathbf{C}^n}$, $G \subset \mathbf{C}^k$ is called analytic if for every plurisubharmonic function $\psi(\lambda, z)$, $\lambda = (\lambda_1, \dots, \lambda_k)$, $z = (z_1, \dots, z_n)$, defined in a neighbourhood of the graph $\{(\lambda, z): \lambda \in G, z \in K_\lambda\}$ the function $\varphi(\lambda) = \max\{\psi(\lambda, z): z \in K_\lambda)$ is plurisubharmonic in G (Slodkowski [8, Def. 4.1]). For $k = n = 1$ this definition is equivalent to that of Oka by [8, Theorem 3.2]. Now by [8, Proposition 5.1] if $\lambda \to K_\lambda : G \to 2^{\mathbf{C}^n}$ and $\lambda \to L_\lambda : G \to 2^{\mathbf{C}^m}$ are analytic multifunctions, then $\lambda \to K_\lambda \times L_\lambda : G \to 2^{\mathbf{C}^{n+m}}$ is also analytic. Therfore, if $\lambda \to K_\lambda: G \to 2^{\mathbf{C}}$ is analytic, then $\lambda \to K_\lambda^n: G \to 2^{\mathbf{C}^n}$ is analytic in the sense of the definition just given, but this is exactly what is stated in Theorem 3.

Several special results known previously can be obtained immediately by means of Theorem 3. Thus if we set $\psi(\lambda, z_1, \dots, z) = \frac{2}{n(n+1)} \sum_{1 \le i < j \le n+1}$.

$\cdot \log|z_i - z_j|$ then we obtain that $\lambda \to \log \delta_n (K_\lambda)$ is subharmonic, where δ_n denotes the *n - th* diameter of a compact plane set. This was obtained by Yamaguchi [11], and, for $K_\lambda = \sigma(T_\lambda)$ and $g(f^{-1}(\lambda))$, by Vesentini and Wermer for $n = 0$, Aupetit and Aupetit-Wermer for $n = 1$ and Senitchkin and Slodkowski for arbitrary n (see [1, 10, 15] for precise references).

Assume now that $E \subset \mathbf{C}$ is open and $\lambda \to K_\lambda : G \to 2^{\mathbf{C}}$ is an analytic multifunction such that $K_\lambda \subset E$, $\lambda \in G$. The Caratheodory pseudodistance c_E on E has the property that the function $(x, y) \to \log c_E(x, y)$ is plurisubharmonic (cf. Vesentini [12] for details). Thus if we apply Theorem 3 to the plurisubharmonic function

$$\psi(\lambda, z_1, \ldots, z_n) = \frac{2}{n(n+1)} \sum_{1 \leq i < j \leq n+1} \log c_E(z_i, z_j),$$

then we obtain that $\lambda \to \log d_n^E(K_\lambda)$ is subharmonic in G, where $d_n^E(K)$ denotes the *n - th* diameter of subset of E defined by means of the Caratheodory pseudodistance c_E. This fact has recently been obtained, in a more direct way, by E. Vesentini [12].

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