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The unbonded contact problem of a rectangular plate resting on an elastic foundation

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Meccanica. — *The unbonded contact problem of a rectangular plate resting on an elastic foundation* (*). Nota di LUIGI ASCIONE (**), RENATO S. OLIVITO (**) e GIUSEPPE SPADEA (**), presentata (***) dal Corrisp. E. GIANGRECO.

RIASSUNTO. — In questo lavoro viene analizzato il problema di equilibrio statico di una piastra rettangolare in contatto unilaterale e senza attrito con un mezzo elastico. Si esaminano i due modelli di fondazione alla Winkler e di semispazio elastico. Il problema viene risolto mediante discretizzazione agli elementi finiti utilizzando un approccio di tipo « penalty ».

La rapida convergenza del metodo e la sua efficienza sono dimostrate dagli esempi studiati, che riguardano sia piastre quadrate che rettangolari sotto diverse condizioni di carico.

NOMENCLATURE

E_p , G_p , ν_p	Plate elastic moduli
χ	Shear correction coefficient
H	Plate thickness
D	Plate flexural stiffness
E_s , ν_s	Half-space elastic moduli
K	Winkler subgrade modulus
$(\cdot)_{,x}$, $(\cdot)_{,y}$	Derivatives with respect to x and y
σ	Dimensionless pressure
V	Dimensionless displacement
Δ	Shear to flexural stiffness ratio
Γ	Plate-half-space (Γ_h) or plate-Winkler subgrade (Γ_w) relative stiffness
$\bar{\varepsilon}$	Penalty parameter.

1. INTRODUCTION

The soil-structure interaction analysis is an interesting problem to both structural and geotechnical engineering. In this analysis the assumptions of linear elastic behaviour of the soil and frictionless bilateral contact between

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structure and subgrade are usually accepted. Nevertheless, different factors such as the load distribution and the plate-soil relative stiffness, can essentially require the hypothesis of unilateral contact.

During recent years unilateral contact problems have been an active subject of research and the use in this framework of penalty formulations, as a basis to develop finite element approximations, has gained much popularity [1–2]. If the main advantages of the penalty approach can be obtained in the case of a rigid obstacle [3–4], this approach is still useful for the contact problem on an elastic obstacle, mainly regarding the convergence aspect. This is shown in [5], where some numerical applications relative to the unilateral contact of a beam or an axisymmetric circular plate on an elastic half-space are presented.

The aim of this paper is to extend the aforementioned analysis to the case of a rectangular plate in frictionless unilateral contact with an elastic foundation. The Winkler subgrade and the isotropic homogeneous half-space models are examined and the results are compared. The scheme of rectangular plate is in fact of great interest in structural engineering and only a few previous numerical investigations on the same subject are available in literature. Mostly, the bonded solutions have been studied and an accurate review of them can be found in the recent text by A.P.S. Selvadurai [6].

For the unbonded case, previous investigations are due to O. J. Svec [7], who analyses the unilateral contact problem of a square plate on an elastic half-space, and to A. Grimaldi and J. N. Reddy [8], who analyse the same problem for an orthotropic plate resting on a Winkler subgrade.

2. FORMULATION OF THE PROBLEM

We consider the static equilibrium problem of a rectangular elastic plate in frictionless unilateral contact with an elastic foundation (fig. 1).

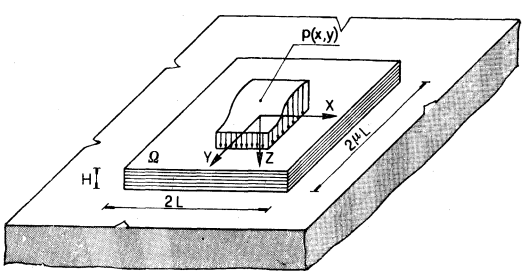


Fig. 1.

The plate displacement field has components:

(2.1a) $$u_x(x, y, z) = -z\psi_x(x, y)$$

(2.1b) $$u_y(x, y, z) = -z\psi_y(x, y)$$

(2.1c) $$u_z(x, y, z) = w(x, y)$$

where ψ_x and ψ_y are the bending slopes along the x and y axes.

Eqs. (2.1), proposed by Mindlin [9], allow us to take into account the effects of the shear stress on the plate deformation. They differ from the corresponding equations in Kirchhoff's theory, because the functions ψ_x and ψ_y substitute the derivatives $w_{,x}$ and $w_{,y}$ of w in the definititions of u_x and u_y (i.e. we assume that the plane sections remain plane after deformation but not necessarily normal to the midplane).

Let be now:

$H^s(\Omega)$: Sobolev space of order s (s is a real positive number);

$(H^s(\Omega))'$: dual space of $H^s(\Omega)$;

$V = H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega)$: space of the plate admissible displacements (ψ_x, ψ_y, w);

$B[(\psi_x, \psi_y, w), (\phi_x, \phi_y, u)]$: bilinear form on V associate to the plate strain energy:

(2.2) $$B(\psi_x, \psi_y, w), (\phi_x, \phi_y, u)] = D \iint_\Omega \Big[\psi_{x,x}\phi_{x,x} + \psi_{y,y}\phi_{y,y} + \nu_p(\psi_{x,x}\phi_{y,y} + \psi_{y,y}\phi_{x,x}) + \frac{1-\nu_p}{2}(\psi_{x,y} + \psi_{y,x})(\phi_{x,y} + \phi_{y,x})\Big]\, dx\, dy + \chi G_p H \iint_\Omega [(w_{,x} - \psi_x)(u_{,x} - \phi_x) + (w_{,y} - \psi_y)(u_{,y} - \phi_y)]\, dx\, dy$$

V_s : space of the admissible displacements w_s relative to the part Ω of the elastic foundation;

V'_s : dual space of V_s; the pressures r, exerted by the plate on the elastic foundation, belong to this space;

$T : V'_s \to V_s$: linear operator which gives the displacements w_s as a function of the pressures r;

$\mathbf{K}' = \{r \in V'_s : \langle r, w_s \rangle \geq 0$

$\forall w_s \in V_s : w_s \geq 0\}$: positive cone in V_s' to which the pressures r in the unilateral contact problem must belong;

p : vertical load acting on the plate; it can be characterized as an element of $(H^1(\Omega))'$;

$\langle \cdot , \cdot \rangle$: duality pairing;

$\| \cdot \|$: norm symbol.

The bilinear from B $[\cdot,\cdot]$ is continuous and positive semi-definite on V, i.e.:

$$\text{(2.3)} \qquad |\, B\,[(\psi_x , \psi_y , w) , (\phi_x , \phi_y , u)]\,| \leq C \,\| (\psi_x , \psi_y , w)\|_V \,\| (\phi_x , \phi_y , u)\|_V$$

$$C \geq 0 , \qquad \forall\, (\psi_x , \psi_y , w) , (\phi_x , \phi_y , u) \in V$$

$$\text{(2.4)} \qquad B\,[(\psi_x , \psi_y , w) , (\phi_x , \phi_y , w)] \geq 0 \qquad \forall\, (\psi_x , \psi_y , w) \in V .$$

The norm $\| \cdot \|_V$ on the space V is definite as:

$$\text{(2.5)} \qquad \| (\psi_x , \psi_y , w)\|_V^2 = \| \psi_x \|^2_{H^1(\Omega)} + \| \psi_y \|^2_{H^1(\Omega)} + \| w \|^2_{H^1(\Omega)} .$$

If the foundation is an isotropic homogeneous elastic half-space, the operator T can be easily expressed in an integral form by using the Green function relative to the classical Boussinesq solution for a vertical point force:

$$\text{(2.6)} \qquad w_s(x , y) = \frac{1 - \nu_s^2}{\pi E_s} \iint_\Omega \frac{r(x', y')}{[(x - x')^2 + (y - y')^2]^{1/2}} \, dx'\, dy' .$$

Eq. (2.6) is valid if the pressure r is sufficiently regular. More generally, the functions w_s are the restrictions to Ω of the vertical displacements. Consequently, they can be defined as elements of the fractional Sobolev space $H^{1/2}(\Omega)$.

From the general properties of the linear elastic problem, we can assume that the operator T is continuous and coercive from V_s' onto V_s, i.e.:

$$\text{(2.7a)} \qquad \| Tr \|_{V_s} \leq C' \, \| r \|_{V_s'}$$

$$\text{(2.7b)} \qquad \langle r , Tr \rangle_{V_s} \geq C'' \, \| r \|^2_{V_s'} \qquad C', C'' \geq 0 \quad , \quad \forall r \in V_s' .$$

If the foundation is modelled as a Winkler subgrade, we have:

$$\text{(2.8)} \qquad T = \frac{1}{K}\, I \quad ; \quad V_s = V_s' = H^0(\Omega)$$

where K is the Winkler modulus and I is the identity operator on the space $H^0(\Omega)$ (the space of the square summable functions on Ω). Properties (2.7) are in this case trivially verified.

Finally, the contact problem can be put in the form:

" Find $(\psi_x, \psi_y, w) \in V$ and $r \in K'$ such that:

$$B[(\psi_x, \psi_y, w), (\phi_x, \phi_y, u)] = \langle p - r, u\rangle_{H^1(\Omega)}, \qquad \forall (\phi_x, \phi_y, u) \in V \tag{2.9a}$$

$$Tr - w \geq 0 \tag{2.9b}$$

$$\langle r, Tr - w\rangle_{V_s} = 0 \text{''}. \tag{2.9c}$$

Eq. (2.9a) represents the set of the internal and boundary equilibrium equations of the plate (virtual work equation), while eqs. (2.9b) and (2.9c) characterize the conditions of unilateral contact.

3. Variational formulation and penalty approach

Let F be the functional:

$$F(\psi_x, \psi_y, w, r) = \tfrac{1}{2} B[(\psi_x, \psi_y, w), (\psi_x, \psi_y, w)] - \tfrac{1}{2}\langle r, Tr\rangle_{V_s} + \langle r, w\rangle_{V_s} - \langle p, w\rangle_{H^1(\Omega)} \tag{3.1}$$

defined on the convex set $V \times \mathbf{K}'$.

It is easy to verify that problem (2.9) is equivalent to the stationary condition of the saddle functional F:

$$DF(\psi_x, \psi_y, w, r; \phi_x, \phi_y, u, s - r) \leq 0 \qquad \forall (\phi_x, \phi_y, u, s) \in V \times \mathbf{K}' \tag{3.2}$$

where $DF(\cdot\,;\,\cdot)$ is the weak differential of F in (ψ_x, ψ_y, w, r) with increment $(\phi_x, \phi_y, u, s - r)$.

The difficulties related to the constraint $r \in \mathbf{K}'$ $(r \geq 0)$ can be avoided by means of a penalty approach. More precisely, we define the external penalty functional:

$$Q(r) = \tfrac{1}{2}\|r^-\|^2_{V'_s} \tag{3.3}$$

where r^- is the negative part of r, which, in the case of the elastic half-space, can be defined as the projection of r on the negative cone in V'_s.

Further on, let F_ε be the perturbation of F:

$$\varepsilon > 0 \quad , \quad F_\varepsilon(\psi_x, \psi_y, w, r) = F(\psi_x, \psi_y, w, r) - \frac{1}{\varepsilon} Q(r). \tag{3.4}$$

An approximate solution of the contact problem can be obtained as a solution of the following problem:

" Find $(\psi_{x_\varepsilon}, \psi_{y_\varepsilon}, w_\varepsilon, r_\varepsilon) \in V \times V'_s$ such that:

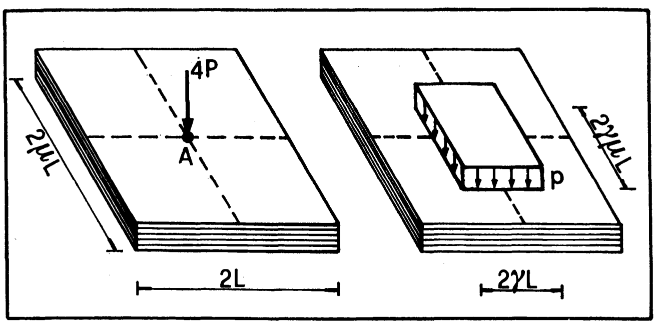
$$\text{(3.5)} \qquad DF_\varepsilon (\psi_{x_\varepsilon}, \psi_{y_\varepsilon}, w_\varepsilon, r_\varepsilon ; \phi_x, \phi_y, u, s) = 0 \qquad \forall (\phi_x, \phi_y, u, s) \in V \times V'_s \text{ ''}.$$

It is possible to show that, if some hypotheses on the external load p are satisfied, problems (3.2) and (3.5) admit a unique solution. Further on, when $\varepsilon \to 0$, the family of approximate solutions $(\psi_{x_\varepsilon}, \psi_{y_\varepsilon}, w_\varepsilon, r_\varepsilon)$ is strongly convergent to the solution of problem (3.2). The mathematical details of the proof can be found in [10].

A review of other variational formulations for problem (2.9) is given in [11].

4. NUMERICAL RESULTS

The variational fomulation given in the previous section is, here, applied to analyse the numerical examples shown in fig. 2.



Load condition 1 Load condition 2

Fig. 2.

In order to deal with dimensionless results, we set:

$$\text{(4.1a)} \qquad s = x/L \quad , \quad t = y/\mu L \quad , \quad R =]-1, +1[\times]-1, +1[$$

$$\text{(4.1b)} \qquad \rho_s = \frac{D}{PL} \psi_x \quad , \quad \rho_t = \frac{D}{PL} \psi_y \quad , \quad V = \frac{D}{PL^2} w$$

$$\text{(4.1c)} \qquad \sigma = \frac{\mu L^2}{P} r \quad , \quad q = \frac{\mu L^2}{P} p$$

$$\Delta = \frac{1}{6\chi(1-\nu_p)}\left(\frac{H}{\mu L}\right)\left(\frac{H}{L}\right) \tag{4.1d}$$

$$\Gamma_h = \frac{1}{12\pi}\,\frac{1-\nu_s^2}{1-\nu_p^2}\,\frac{E_p}{E_s}\left(\frac{H}{\mu L}\right)\left(\frac{H}{L}\right)^2; \tag{4.1e}$$

$$\Gamma_w = \frac{1}{12(1-\nu_p^2)}\,\frac{E_p}{KL}\left(\frac{H}{\mu L}\right)\left(\frac{H}{L}\right)^2$$

$$\bar{\varepsilon} = \frac{\mu L^4}{D}\,\Gamma\varepsilon \tag{4.1f}$$

where $4\,P$ is the resultant of the external load acting on the plate.

The parameters Δ and Γ (Γ_h or Γ_w) represent, respectively, the ratio of the plate shear stiffness to the flexural one and the ratio of the plate stiffness to the foundation one.

The region R is partitioned into a collection $\{R_e\}_{e=1}^{E}$ $\left(R = \bigcup_{e=1}^{E} R_e,\ e \neq f \Rightarrow R_e \cap R_f = \varnothing\right)$ of rectangular finite elements over which bilinear approximations of the unknown functions are constructed (four node rectangular elements).

From eq. (3.5) by simple algebra we can obtain the following set of equations:

$$\boldsymbol{K}^{(s,s)}\,\varrho_s + \boldsymbol{K}^{(s,t)}\,\varrho_t + \boldsymbol{K}^{(s,v)}\,\boldsymbol{V} = \boldsymbol{0} \tag{4.2a}$$

$$\boldsymbol{K}^{(t,s)}\,\varrho_s + \boldsymbol{K}^{(t,t)}\,\varrho_t + \boldsymbol{K}^{(t,v)}\,\boldsymbol{V} = \boldsymbol{0} \tag{4.2b}$$

$$\boldsymbol{K}^{(v,s)}\,\varrho_s + \boldsymbol{K}^{(v,t)}\,\varrho_t + \boldsymbol{K}^{(v,v)}\,\boldsymbol{V} + \boldsymbol{M}\sigma = \boldsymbol{q} \tag{4.2c}$$

$$\boldsymbol{MV} - \left(\boldsymbol{T} + \frac{1}{\bar{\varepsilon}}\,\Sigma(\sigma)\right)\sigma = \boldsymbol{0} \tag{4.2d}$$

where $\varrho_s, \varrho_t, \boldsymbol{V}, \sigma$ are the vectors of the nodal values and the matrices $\boldsymbol{K}^{(\cdot,\cdot)}$, $\boldsymbol{M}, \boldsymbol{T}, \boldsymbol{\Sigma}, \boldsymbol{q}$ can be easily expressed as functions of the global interpolants $f_i(s, t)$ and of the parameters Δ and Γ. In particular the matrix $\boldsymbol{\Sigma}$, which is a non-linear function of σ, can be expressed as:

$$\Sigma_{ij}(\sigma) = \sum_{e=1}^{E}\ \sum_{\gamma_e=1}^{G_e} \Gamma W_{\gamma_e} f_i(s_{\gamma_e}, t_{\gamma_e}) f_j(s_{\gamma_e}, t_{\gamma_e})\,. \tag{4.3}$$

In eq. (4.3) the sum is extended to all elements E and to all Gauss quadrature points G_e over the element e. The coefficients W_{γ_e} are defined as:

$$W_{\gamma_e} = \begin{cases} P_{\gamma_e}\ \text{(Gaussian weight to the point } (s_{\gamma_e}, t_{\gamma_e})) \\ \quad \text{if}\quad \sigma_i f_i(s_e, \gamma_e) < 0) \\ 0 \quad \text{if}\quad \sigma_i f_i(s_{\gamma_e}, t_{\gamma_e}) \geq 0\,. \end{cases} \tag{4.4}$$

Eqs (4.2) can be solved by means of the following iterative procedure:

$$K^{(s,s)}\,\varrho_s^{(i)} + K^{(s,t)}\,\varrho_t^{(i)} + K^{(s,v)}\,V^{(i)} = 0 \tag{4.5a}$$

$$K^{(t,s)}\,\varrho_s^{(i)} + K^{(t,t)}\,\varrho_t^{(i)} + K^{(t,v)}\,V^{(i)} = 0 \tag{4.5b}$$

$$K^{(v,s)}\,\varrho_s^{(i)} + K^{(v,t)}\,\varrho_t^{(i)} + \bar{K}^{(v,v)}\,V^{(i)} = q \tag{4.5c}$$

where:

$$\bar{K}^{(v,v)} = M\left[T + \frac{1}{\bar{\varepsilon}}\,\Sigma\,(\sigma^{(i-1)})\right]^{-1} M + K^{(v,v)}\,. \tag{4.6}$$

The mechanical meaning of this iterative scheme is immediate: at the i-th step we solve a bilateral problem relative to a system composed by the plate, the elastic foundation and some elastic springs with relative stiffness $\bar{\varepsilon}\,\Gamma$ in respect of the plate; these springs are added at the interface, between plate and soil, where the pressures $\sigma^{(i-1)}$ of the previous step are negative.

The first step corresponds to the bilateral contact between plate and elastic foundation.

Figs. 3–6 show the plots of the dimensionless displacements V and pressures σ along the plate sides and along the diagonal. These results correspond to the values 0.3, 0.002857 and 10^{-5} of the parameters ν_p, Δ ($\chi = 5/6$, $H/L = 0.1$) and $\bar{\varepsilon}$, respectively. Further on, the same values of the parameters Γ_h and Γ_w have been considered in order to compare the behaviours corresponding to the two foundation models analysed: $\Gamma_1 = 0.25 \times 10^{-3}$, $\Gamma_2 = 0.75 \times 10^{-3}$, $\Gamma_3 = 0.25 \times 10^{-2}$ for square plates; $\Gamma_1 = 0.75 \times 10^{-3}$, $\Gamma_2 = 2.25 \times 10^{-3}$, $\Gamma_3 = 0.75 \times 10^{-2}$ for rectangular plates ($\mu = 0.5$, i.e. the ratio between the sides is 0.5).

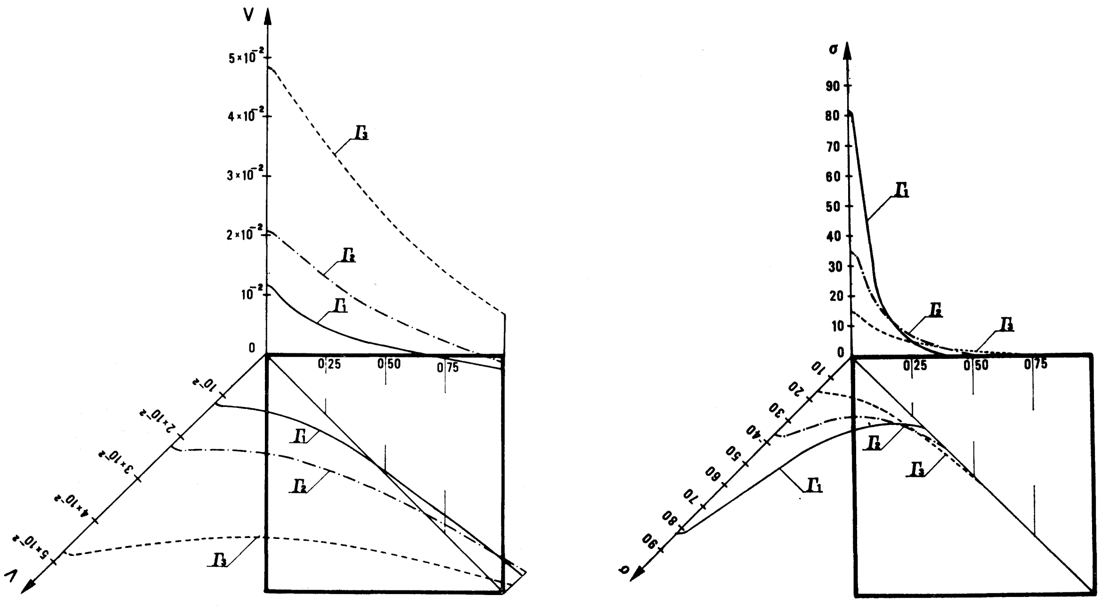
Due to the symmetry of the scheme, only one-quarter of plate has been discretized by a uniform mesh of 8×8 (elastic half-space) or 10×10 finite elements (Winkler subgrade).

The above mentioned figures show as the contact regions are diminishing with the decreasing values of Γ. In particular, for $\Gamma = 0.25 \times 10^{-3}$, the contact radius that we have determined for a square plate resting on an elastic half-space and subject to a point force is 0.44. In the same case, for a Kirchhoff plate, the value of the contact radius found by Svec in [7] is 0.50. Due to lack of other results in literature no further assessment of accuracy can be made.

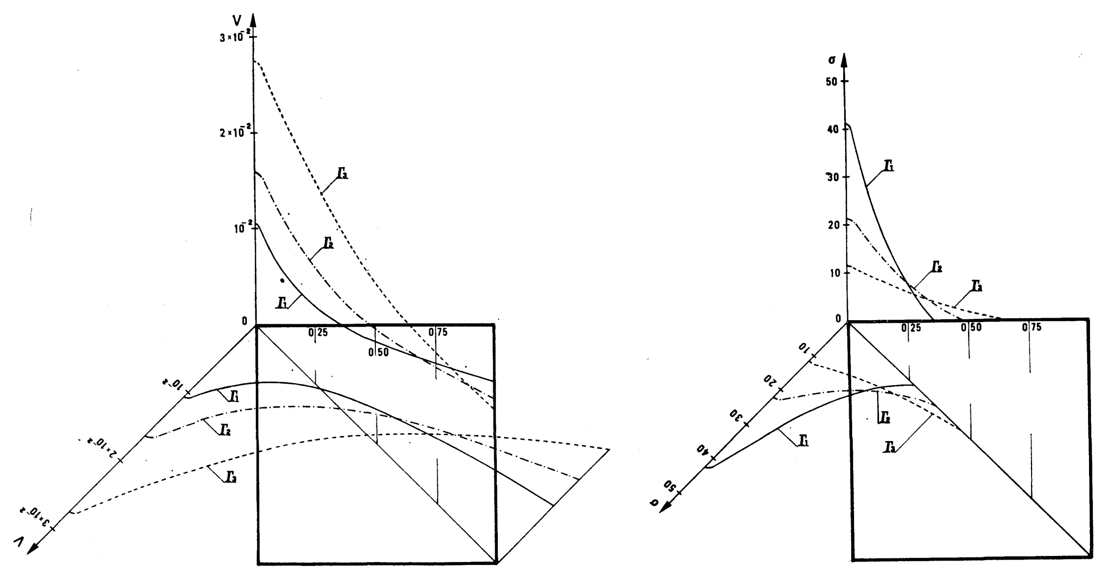
Finally, fig. 7 shows some comparisons between bonded and unbonded solutions for square plates under the load condition 1.

More details relative to the numerical aspects of the problem can be found in [12].

Square plate: loading condition 1



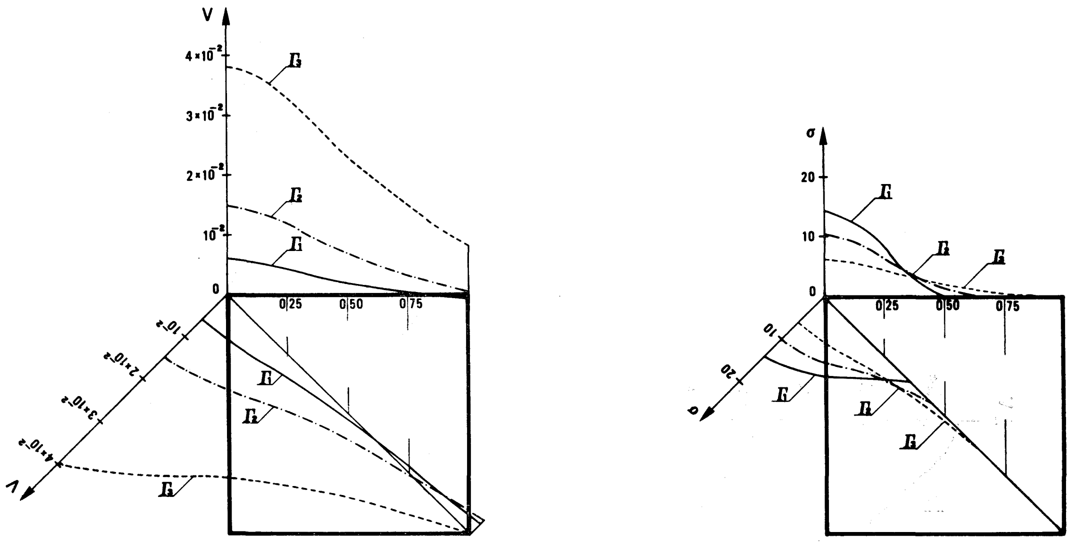
a) Elastic half–space *b*)



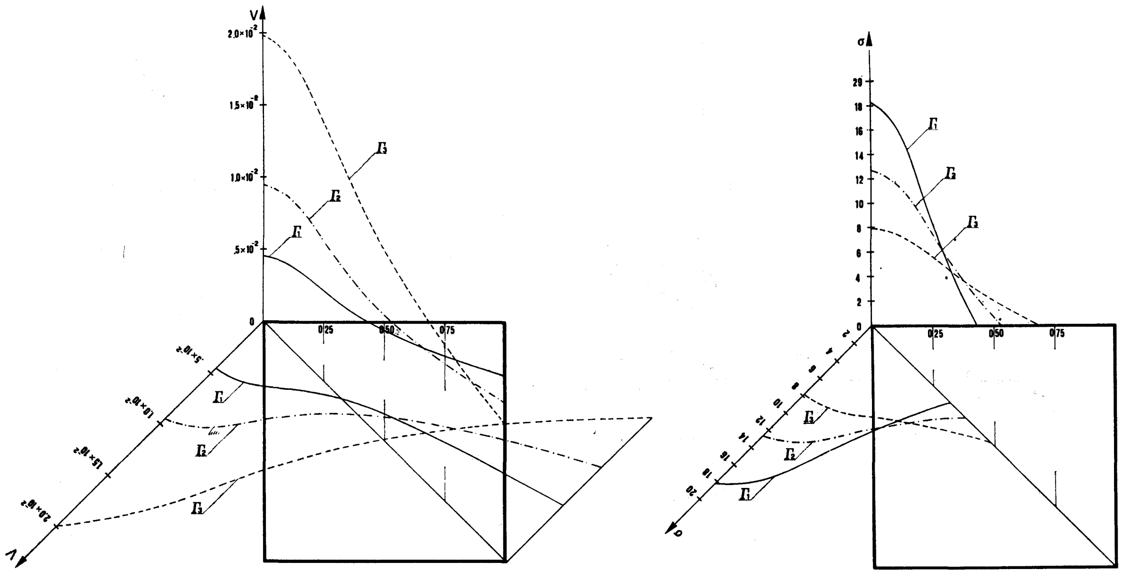
c) Winkler subgrade *d*)

Fig. 3.

Square plate: load condition 2



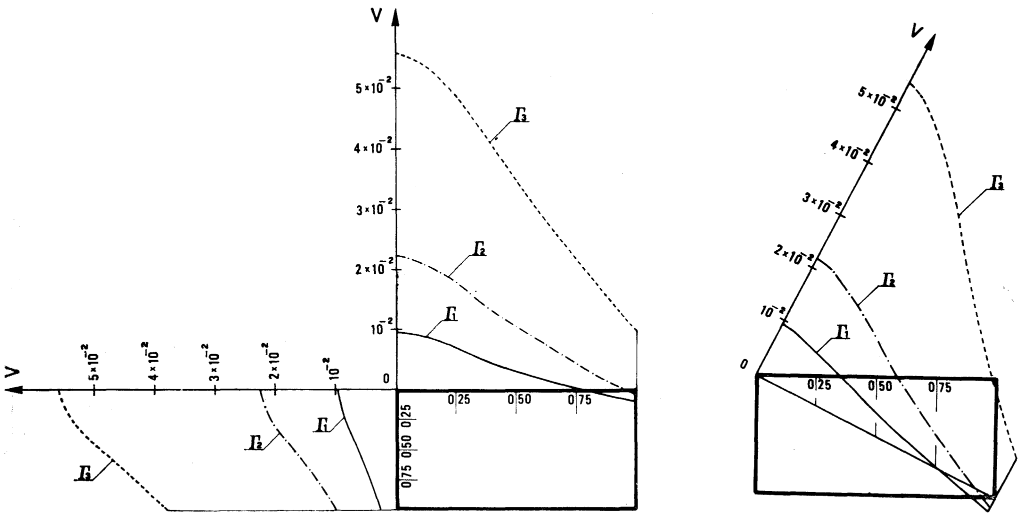
a) Elastic half-space (γ = 0.25) *b*)



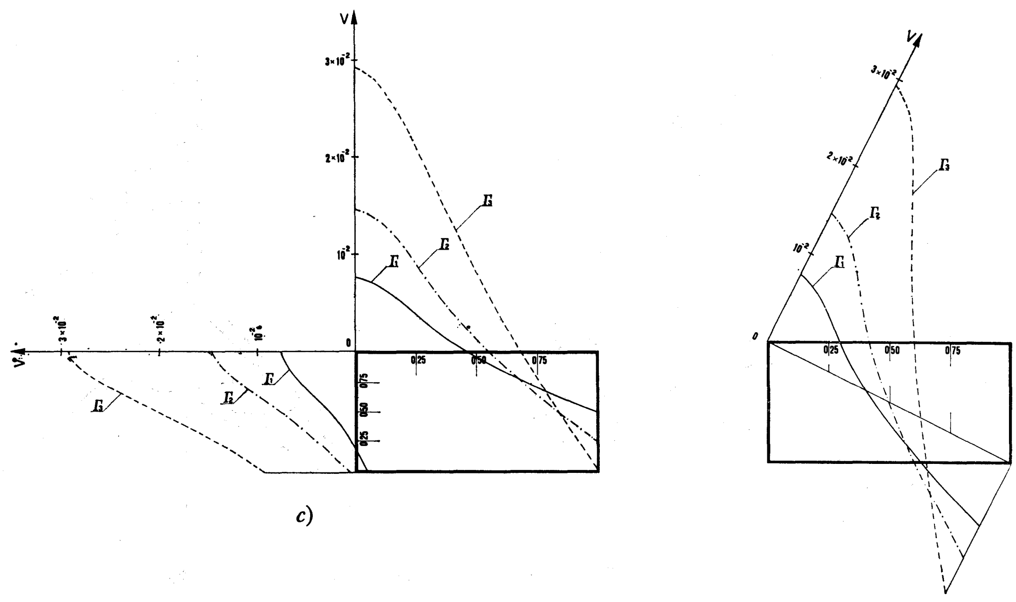
c) Winkler subgrade (γ = 0.20) *d*)

Fig. 4.

Rectangular plate ($\mu = 0.5$): load condition 2



a) Elastic half-space ($\gamma = 0.25$) *b*)

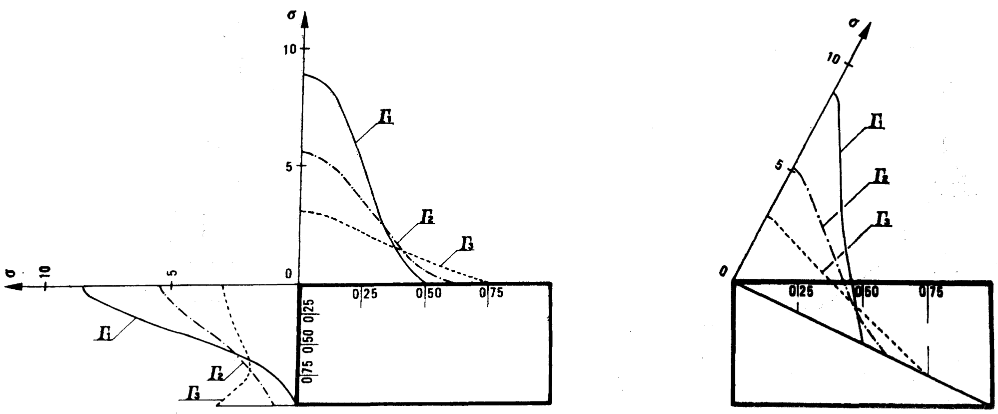


c)

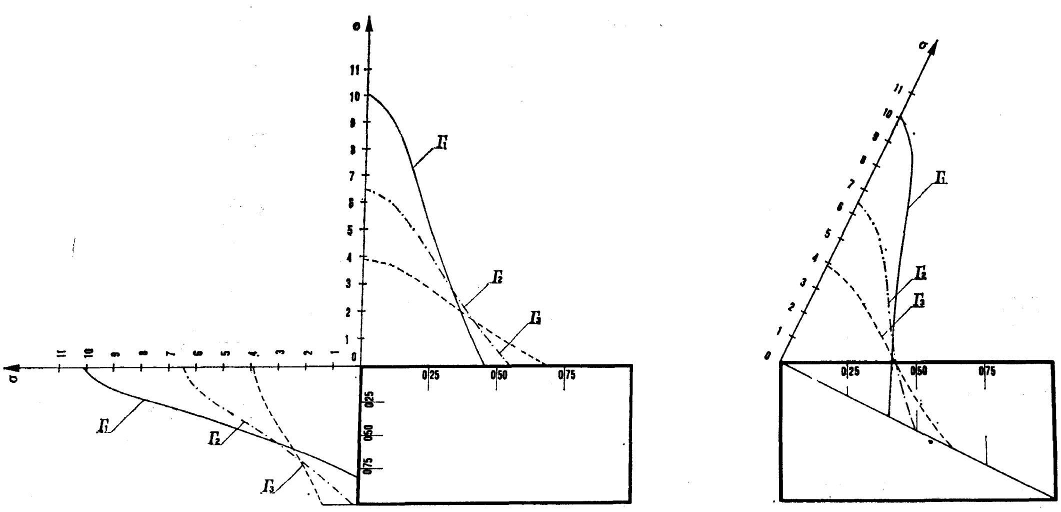
Winkler subgrade ($\gamma = 0.20$) *d*)

Fig. 5.

Rectangular plate ($\mu = 0.5$): load condition 2



a) Elastic half-space ($\gamma = 0.25$) *b*)



c) Winkler subgrade ($\gamma = 0.20$) *d*)

Fig. 6.

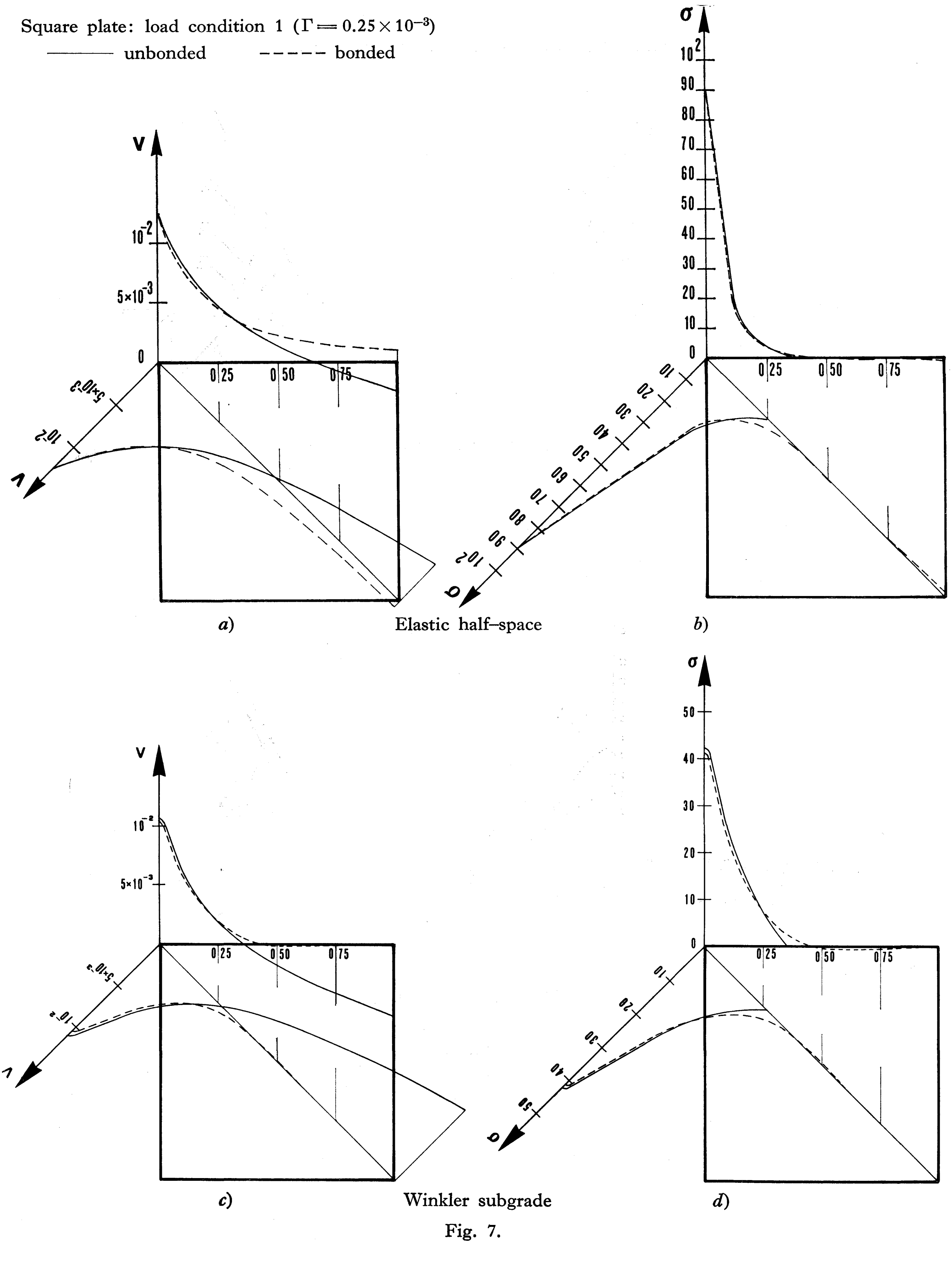

Square plate: load condition 1 ($\Gamma = 0.25 \times 10^{-3}$)
——— unbonded
– – – – – bonded
V
10^{-2}
5×10^{-3}
0
0|25
0|50
0|75
σ
10^{2}
90
80
70
60
50
40
30
20
10
a)
Elastic half–space
b)
c)
Winkler subgrade
d)

Fig. 7.

5. CONCLUSIONS

The previous analysis allows us to make the following conclusions:

i) the iterative procedure that we have utilized to get the discrete solution is very efficient and the convergence is reached in a few steps (only $4 \sim 6$ steps);

ii) the same contact regions occur approximatively in the two cases of elastic half-space or Winkler subgrade if equal values of the relative stiffness parameter Γ_h and Γ_w are considered, although the values of displacements and pressures can be different;

iii) the bonded and unbonded solutions can give completely different results, mainly regarding the displacements.

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