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Enrico Magenes, Claudio Verdi, Augusto Visintin

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Equazioni a derivate parziali. — *Semigroup approach to the Stefan problem with non-linear flux.* Nota (*) di ENRICO MAGENES (**), CLAUDIO VERDI (***) e AUGUSTO VISINTIN (****), presentata dal Corrisp. E. MAGENES.

RIASSUNTO. — Un problema di Stefan a due fasi con condizione di flusso non lineare sulla parte fissa della frontiera è affrontato mediante la teoria dei semigruppi di contrazione in L^1. Si dimostra l'esistenza e l'unicità della soluzione nel senso di Crandall-Liggett e Bénilan.

Here we study the two-phase Stefan problem in more space variables with a non-linear flux condition on the fixed boundary. Denoting the space domain by Ω and the enthalpy density by u, we have a problem of the form

$$(\mathrm{P}) \quad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta \beta(u) = f & \text{in } \Omega \times]0, T[\\ \dfrac{\partial \beta(u)}{\partial \nu} + g(\beta(u)) = 0 & \text{on } \partial\Omega \times]0, T[\\ u(0) = u_0 & \text{in } \Omega ; \end{cases}$$

the non-decreasing function β is characteristic of the material, $\beta(u)$ represents the temperature, f is a datum and g is a given (in general *non-linear*) function, as for the classical Stefan-Boltzmann radiation law.

Following the classical variational formulation in $L^2(\Omega)$ (for a discussion and further references see [12], e.g.), problem (P) has been recently studied in [5, 14, 15]. Here we use an approach based on the theory of non-linear contraction semigroups in $L^1(\Omega)$, following ideas and techniques used

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(**) Dipartimento di Matematica dell'Università di Pavia e Istituto di Analisi Numerica del C.N.R. di Pavia.

(***) Istituto di Matematica, Informatica e Sistemistica della Universita di Udine.

(****) Istituto di Analisi Numerica del C.N.R. di Pavia.

for similar problems in [2, 3, 4, 7, 8, 9]. We show that the operator $Aw = -\Delta\beta(w)$ with domain

$$D(A) = \{w \in L^1(\Omega) \mid \beta(w) \in W^{1,1}(\Omega)\ ,\ \Delta\beta(w) \in L^1(\Omega)\ ,\ \frac{\partial\beta(w)}{\partial\nu} + \\ + g(\beta(w)) = 0 \text{ on } \Gamma\}$$

generates a contraction semigroup in $L^1(\Omega)$; this yields the existence and uniqueness of the generalized solution of problem (P) in the sense of Crandall-Liggett and Bénilan. This approach seems especially useful for the numerical solution (see [3, 13]).

§ 1. THE CASE OF NO INTERNAL SOURCE ($f = 0$)

Let $\Omega \subset \mathbf{R}^N$ be a bounded regular domain for instance of class C^∞, with boundary Γ. Let

$$(1) \quad \begin{cases} \beta : \mathbf{R} \to \mathbf{R} \text{ Lipschitz-continuous and non-decreasing, } \beta(0) = 0 \\ |\beta(\xi)| \geqq C_1 |\xi| - C_2 \quad \forall \xi \in \mathbf{R}\ (C_1, C_2 : \text{positive constants}) \end{cases}$$

(it is not restrictive to assume that the Lipschitz-constant of β is 1)

$$(2) \quad \begin{cases} g \in C^1(\mathbf{R}) \text{ non-decreasing, } g(0) = 0 \\ |g(\xi)| \leqq C_3 |\xi| + C_4 \quad \forall \xi \in \mathbf{R}\ (C_3, C_4 : \text{positive constants}) \end{cases}$$

(an explicit dependence of g on $\sigma \in \Gamma$ would cause no further difficulty).

We introduce the non-linear operator $A : w \to \Delta\beta(w)$ with domain

$$D(A) = \{w \in L^1(\Omega) \mid \beta(w) \in W^{1,1}(\Omega)\ ,$$

$$\Delta\beta(w) \in L^1(\Omega) \quad \text{and } \frac{\partial\beta(w)}{\partial\nu} + g(\beta(w)) = 0 \text{ on } \Gamma\}\ .$$

Here the trace $\beta(w)$ and the external normal trace $\dfrac{\partial\beta(w)}{\partial\nu}$ are understood in the sense of Gagliardo (see [10] e.g.) and are in $L^1(\Gamma)$; by the growth assumption on g, also $g(\beta(w)) \in L^1(\Gamma)$. The condition on Γ can also be written in the form

$$(3) \quad \int_\Omega \nabla\beta(w) \cdot \nabla v \, dx + \int_\Gamma g(\beta(w)) \cdot v d\sigma = -\int_\Omega \Delta\beta(w) \cdot v dx \ \ \forall v \in C^1(\bar{\Omega})\ .$$

THEOREM 1. *A is m-accretive in* $L^1(\Omega)$, *that is*

$$(4)\quad \begin{cases} \forall f \in L^1(\Omega)\ ,\ \forall\lambda > 0\ ,\ \exists\, !\, w \in D(A) \text{ such that} \\ w - \lambda\Delta\beta(w) = f \quad \text{a.e. in } \Omega\,,\ \text{i.e.} \\ \int\limits_\Omega w\cdot v\,dx + \lambda\int\limits_\Omega \nabla\beta(w)\cdot\nabla v dx + \lambda\int\limits_\Gamma g(\beta(w))\cdot v\,d\sigma = \int\limits_\Omega f\cdot v dx\ \forall v \in C^1(\bar\Omega), \end{cases}$$

$$(5)\quad \forall\lambda > 0\,,\ (I + \lambda A)^{-1} \text{ is a contraction in } L^1(\Omega)\ (I \equiv \text{Identity})\,.$$

Proof. This is split into several steps.

(i) *Uniqueness of the solution of* (4).

Let w_1, w_2 be two solutions; setting $\theta_i = \beta(w_i)$ $(i = 1, 2)$ we have

$$(6)\quad \theta_i - \lambda\Delta\theta_i = f - w_i + \theta_i \equiv \Phi_i \qquad \text{in } \Omega$$

$$(7)\quad -\frac{\partial\theta_i}{\partial v} = g(\theta_i) \equiv \psi_i \qquad \text{on } \Gamma\,.$$

Let $\{\Phi_{i,n} \in L^2(\Omega)\}_{n\in N}$, $\{\psi_{i,n} \in H^{1/2}(\Gamma)\}_{n\in N}$ be such that $\Phi_{i,n} \to \Phi_i$ strongly in $L^1(\Omega)$, $\psi_{i,n} \to \psi_i$ strongly in $L^1(\Gamma)$; by well-known results (see [11], e.g.), the elliptic problem

$$(8)\quad \theta_{i,n} - \lambda\Delta\theta_{i,n} = \Phi_{i,n} \qquad \text{in } \Omega$$

$$(9)\quad -\frac{\partial\theta_{i,n}}{\partial\nu} = \psi_{i,n} \qquad \text{on } \Gamma$$

has one and only one solution $\theta_{i,n} \in H^2(\Omega)$. By Lemma 2.3 of [4] we have

$$(10)\quad \|\theta_i - \theta_{i,n}\|_{W^{1,1}(\Omega)} \leqq C\,(\|\Phi_{i,n} - \Phi_i\|_{L^1(\Omega)} + \|\psi_{i,n} - \psi_i\|_{L^1(\Gamma)})\,,$$

with C constant independent of i, n; therefore

$$(11)\quad \theta_{i,n} \to \theta_i \quad \text{strongly in } W^{1,1}(\Omega) \quad \text{as } n \to \infty\,.$$

We approximate the Heaviside graph H as follows

$$(12)\quad \{H_j \in C^1(\mathbf{R})\}_{j\in N}\,,\quad H_j' \geqq 0\,,\quad H_j(\xi) = 0 \text{ for } \xi \leqq 0\,,$$
$$H_j(\xi) = 1 \text{ for } \xi \geqq \frac{1}{j}\,.$$

Taking the difference between (8) written for $i=1,2$ and multiplying by $H_j(\theta_{1,n}-\theta_{2,n})$, we get

$$(13)\quad \int_\Omega (\theta_{1,n}-\theta_{2,n})\cdot H_j(\theta_{1,n}-\theta_{2,n})\,dx+\lambda\int_\Omega \nabla(\theta_{1,n}-\theta_{2,n})\cdot\nabla H_j(\theta_{1,n}-\theta_{2,n})\,dx+\lambda\int_\Gamma(\psi_{1,n}-\psi_{2,n})\cdot H_j(\theta_{1,n}-\theta_{2,n})\,d\sigma=\int_\Omega(\Phi_{1,n}-\Phi_{2,n})\cdot H_j(\theta_{1,n}-\theta_{2,n})\,dx;$$

as $H_j'\geqq 0$, the second integral is non-negative; we can assume that the sequences $\{\Phi_{i,n}\}$ and $\{\psi_{i,n}\}$ are dominated by integrable functions for $i=1,2$; thus taking $n\to\infty$ in (13) we get

$$\int_\Omega(\theta_1-\theta_2)\cdot H_j(\theta_1-\theta_2)\,dx+\lambda\int_\Gamma[g(\theta_1)-g(\theta_2)]\cdot H_j(\theta_1-\theta_2)\,dx\leqq$$

$$\leqq\int_\Omega(\Phi_1-\Phi_2)\cdot H_j(\theta_1-\theta_2)\,dx=\int_\Omega[(\theta_1-\theta_2)-(w_1-w_2)]\cdot H_j(\theta_1-\theta_2)\,dx\,.$$

The second integral is non-negative by the monotonicity of g and the second member is non-positive by the properties of β; thus taking $j\to\infty$ we get

$$\int_\Omega(\theta_1-\theta_2)^+\,dx\leqq 0\,.$$

Interchanging θ_1 and θ_2 we have $\theta_1=\theta_2$ a.e. in Ω, whence by (6) $w_1=w_2$ a.e. in Ω.

(ii) $\forall f\in L^2(\Omega)$, $\forall\lambda>0$, $\exists\, w\in D(A)$ *solution of* (4).

Using a standard procedure, we approach β and g by two sequences described by a positive parameter ε as follows

$$\beta_\varepsilon\in C^\infty(\mathbf{R})\ ,\ 0<\varepsilon\leqq\beta'\leqq 1\ ,\qquad \beta_\varepsilon(0)=0\ ,\ \beta_\varepsilon\to\beta \text{ uniformly in } \mathbf{R}$$

$$g_\varepsilon\in C^\infty(\mathbf{R})\ ,\ g_\varepsilon'\geq 0\ ,\ g_\varepsilon(0)=0\ ,\ g_\varepsilon\to g \quad\text{uniformly in } \mathbf{R};$$

we also assume that β_ε is uniformly Lipschitz-continuous and that g_ε fulfills an order of growth assumption as in (2); moreover let

$$(14)\qquad f_\varepsilon\in C^\infty(\mathbf{R})\ ,\ f_\varepsilon\to f\quad\text{strongly in } L^2(\Omega)\,.$$

We consider the ε-regularized problems corresponding to (4); setting $\theta_\varepsilon \equiv \beta_\varepsilon(w_\varepsilon)$, $R_\varepsilon \equiv \beta_\varepsilon^{-1} - I$, this can be written also in the form

$$\theta_\varepsilon - \lambda\Delta\theta_\varepsilon + R_\varepsilon(\theta_\varepsilon) = f_\varepsilon \qquad \text{in } \Omega \tag{15}$$

$$\frac{\partial\theta_\varepsilon}{\partial\nu} + g_\varepsilon(\theta_\varepsilon) = 0 \qquad \text{on } \Gamma; \tag{16}$$

by well-known results (see [10], e.g.), this problem has one and only one solution $\theta_\varepsilon \in C^1(\bar{\Omega})$, for instance. Multiplying (15) by θ_ε, by a standard procedure we get the a priori estimate

$$\| \theta_\varepsilon \|_{H^1(\Omega)} \leqq C_\lambda \qquad \text{(constant dependent on } \lambda \text{ but not on } \varepsilon) , \tag{17}$$

whence $\| \theta_\varepsilon \|_{L^2(\Gamma)} \leqq C_\lambda$ and by the assumptions on g_ε

$$\| g_\varepsilon(\theta_\varepsilon) \|_{L^2(\Gamma)} \leqq C_\lambda ; \tag{18}$$

by the assumptions on β and β_ε , (17) entails also

$$\| w_\varepsilon \|_{L^2(\Omega)} \leqq C_\lambda . \tag{19}$$

By the previous a priori estimates there exist w , θ , η such that, possibly taking subsequences, as $\varepsilon \to 0$

$$w_\varepsilon \to w \qquad \text{weakly in } L^2(\Omega) \tag{20}$$

$$\theta_\varepsilon \equiv \beta_\varepsilon(w_\varepsilon) \to \theta \qquad \text{weakly in } H^1(\Omega) \tag{21}$$

$$g_\varepsilon(\beta_\varepsilon(w_\varepsilon)) \to \eta \qquad \text{weakly in } L^2(\Gamma) . \tag{22}$$

Using standard monotonicity techniques, one can show that

$$\theta = \beta(w) \quad \text{a.e. in } \Omega , \quad \eta = g(\beta(w)) \quad \text{a.e. on } \Gamma , \tag{23}$$

therefore taking $\varepsilon \to 0$ in (15), (16) a solution of (4) is obtained with the further regularity

$$w \in L^2(\Omega) \quad , \quad \beta(w) \in H^1(\Omega) \quad , \quad \Delta\beta(w) \in L^2(\Omega) .$$

(iii) $\forall\lambda > 0$, $(I + \lambda A)^{-1}$: $L^2(\Omega) \to L^2(\Omega)$ *is a contraction with respect to the norm of* $L^1(\Omega)$;
i.e. for any $f_1 , f_2 \in L^2(\Omega)$, denoting the corresponding solutions of (4) by w_1 , w_2 , we have

$$\| w_1 - w_2 \|_{L^1(\Omega)} \leqq \| f_1 - f_2 \|_{L^1(\Omega)} . \tag{24}$$

In order to prove this, we consider $f_{1,\varepsilon}, f_{2,\varepsilon}$ as in (14) and denote the corresponding solutions of (15), (16) by $w_{1,\varepsilon}, w_{2,\varepsilon}$. Taking the difference between (15) written for $i=1,2$ and multiplying by $H_j(\theta_{1,\varepsilon}-\theta_{2,\varepsilon})$, we get

$$\int_\Omega (w_{1,\varepsilon}-w_{2,\varepsilon})\cdot H_j(\theta_{1,\varepsilon}-\theta_{2,\varepsilon})\,dx + \lambda\int_\Omega \nabla(\theta_{1,\varepsilon}-\theta_{2,\varepsilon})\cdot\nabla H_j(\theta_{1,\varepsilon}-\theta_{2,\varepsilon})\,dx +$$

$$+\lambda\int[g_\varepsilon(\theta_{1,\varepsilon})-g_\varepsilon(\theta_{2,\varepsilon})]\cdot H_j(\theta_{1,\varepsilon}-\theta_{2,\varepsilon})\,d\sigma =$$

$$=\int_\Omega (f_{1,\varepsilon}-f_{2,\varepsilon})\cdot H_j(\theta_{1,\varepsilon}-\theta_{2,\varepsilon})\,dx,$$

whence, as the second and third integrals are non-negative,

$$\int_\Omega (w_{1,\varepsilon}-w_{2,\varepsilon})\cdot H_j(\theta_{1,\varepsilon}-\theta_{2,\varepsilon})\,dx \leqq \int_\Omega (f_{1,\varepsilon}-f_{2,\varepsilon})\cdot H_j(\theta_{1,\varepsilon}-\theta_{2,\varepsilon})\,dx \leqq \tag{25}$$

$$\leqq \|f_{1,\varepsilon}-f_{2,\varepsilon}\|_{L^1(\Omega)}.$$

Note that, denoting the Heaviside graph by H, there exists $\chi\in H(\theta_{1,\varepsilon}-\theta_{2,\varepsilon})$ such that

$$H_j(\theta_{1,\varepsilon}-\theta_{2,\varepsilon})\to\chi \qquad \text{weakly star in } L^\infty(\Omega);$$

by the strict monotonicity of β_ε we have also $\chi\in H(w_{1,\varepsilon}-w_{2,\varepsilon})$, hence taking $j\to\infty$ in (25) we get

$$\int_\Omega (w_{1,\varepsilon}-w_{2,\varepsilon})^+ + dx \leqq \|f_{1,\varepsilon}-f_{2,\varepsilon}\|_{L^1(\Omega)} \tag{26}$$

Interchanging $w_{1,\varepsilon}$ and $w_{2,\varepsilon}$ we have

$$\int_\Omega (w_{2,\varepsilon}-w_{1,\varepsilon})^+\,dx \leqq \|f_{1,\varepsilon}-f_{2,\varepsilon}\|_{L^1(\Omega)} \tag{27}$$

and then taking $\varepsilon\to 0$ in (26), (27) we get (24).

(iv) $\forall f\in L^1(\Omega)$, $\forall\lambda>0$, $\exists w\in D(A)$ *such that* $w-\lambda\Delta\beta(w)=f$ *a.e. in* Ω.

Let $\{f_n\in L^2(\Omega)\}_{n\in N}$, $f_n\to f$ strongly in $L^1(\Omega)$; denote by w_n the solution of (4) corresponding to f_n. Thus, setting $\theta_n\equiv\beta(w_n)$, $\theta_n\in\{\theta\in W^{1,1}(\Omega)\mid \Delta\theta\in L^1(\Omega)$, $\frac{\partial\theta}{\partial\nu}+g(\theta)=0$ on $\Gamma\}$ and

$$\theta_n-\lambda\Delta\theta_n=f_n-w_n+\theta_r \quad \text{in } \Omega. \tag{28}$$

By (iii) $\{w_n\}_{n\in N}$ is a Cauchy sequence in $L^1(\Omega)$, thus there exists $w \in L^1(\Omega)$ such that

$$w_n \to w \qquad \text{strongly in } L^1(\Omega),$$

whence, as β is Lipschitz-continuous, also

$$\theta_n = \beta(w_n) \to \theta = \beta(w) \qquad \text{strongly in } L^1(\Omega);$$

therefore

$$f_n - w_n + \theta_n \to f - w + \theta \qquad \text{strongly in } L^1(\Omega)$$

and taking $n \to \infty$ in (28) we get that w solves (4) since $-\Delta$ is m-accretive in $L^1(\Omega)$ with domain D (see [4], e.g).

(v) $\forall\lambda > 0$, $(I + \lambda A)^{-1}$ *is a contraction in* $L^1(\Omega)$,

i.e. $\forall f_1, f_2 \in L^1(\Omega)$, denoting the corresponding solutions of (4) by w_1, w_2,

(29) $$\| w_1 - w_2 \|_{L^1(\Omega)} \leqq \| f_1 - f_2 \|_{L^1(\Omega)} .$$

In order to prove this, let $\{f_{i,n} \in L^2(\Omega)\}_{n\in N}$, $f_{i,n} \to f_i$ strongly in $L^1(\Omega)$ $(i = 1, 2)$; let $w_{i,n}$ denote the solution of (4) corresponding to $f_{i,n}$. As we proved in (iv)

$$w_{i,n} \to w_i \qquad \text{strongly in } L^1(\Omega);$$

by (iii)

$$\| w_{1,n} - w_{2,n} \|_{L^1(\Omega)} \leqq \| f_{1,n} - f_{2,n} \|_{L^1(\Omega)}$$

and taking $n \to \infty$ we get (29).

THEOREM 2. D (A) *is dense in* $L^1(\Omega)$.

Proof. As

$$D(A)_2 \equiv \{w \in L^2(\Omega) \mid \beta(w) \in H^1(\Omega)\ ,\ \Delta\beta(w) \in L^2(\Omega)\,,\ \frac{\partial\beta(w)}{\partial\nu} + g(\beta(w)) = 0 \text{ on } \Gamma\} \subset D(A)$$

and the inclusion $\mathfrak{D}(\Omega) \subset L^1(\Omega)$ is dense, it is sufficient to prove that

$$\forall f \in \mathfrak{D}(\Omega)\,, \qquad \text{setting } w_\lambda = (I + \lambda A)^{-1} f \quad \text{with } w \in D(A)_2\,,$$

then

$$w_\lambda \to f \quad \text{strongly in } L^2(\Omega) \text{ as } \lambda \to 0,$$

or equivalently

$$\lambda \Delta \beta(w_\lambda) \to 0 \quad \text{strongly in } L^2(\Omega). \tag{30}$$

To this aim we consider the regularized problems in $\beta_\varepsilon, g_\varepsilon, f_\varepsilon = f$ with solutions $w_{\lambda,\varepsilon}$ and we multiply the corresponding equation (15) by $-\Delta\beta_\varepsilon(w_{\lambda,\varepsilon})$, getting

$$\int_\Omega \nabla w_{\lambda,\varepsilon} \cdot \nabla\beta_\varepsilon(w_{\lambda,\varepsilon})\, dx + \int_\Gamma g_\varepsilon(\beta_\varepsilon(w_{\lambda,\varepsilon})) \cdot w_{\lambda,\varepsilon}\, d\sigma +$$

$$+ \lambda \int_\Omega [\Delta\beta_\varepsilon(w_{\lambda,\varepsilon})]^2\, dx = \int_\Omega \nabla f \cdot \nabla\beta_\varepsilon(w_{\lambda,\varepsilon})\, dx \leqq \| \nabla f \|_{L^2(\Omega)} \cdot$$

$$\| \nabla\beta_\varepsilon(w_{\lambda,\varepsilon}) \|_{L^2(\Omega)}.$$

As $g_\varepsilon(0) = \beta_\varepsilon(0) = 0$ and $g_\varepsilon, \beta_\varepsilon$ are monotone, the second integral is non-negative; moreover, by the properties of β_ε,

$$\int_\Omega \nabla w_{\lambda,\varepsilon} \cdot \nabla\beta_\varepsilon(w_{\lambda,\varepsilon})\, dx \geq \int_\Omega | \nabla\beta_\varepsilon(w_{\lambda,\varepsilon}) |^2\, dx;$$

hence

$$\| \nabla\beta_\varepsilon(w_{\lambda,\varepsilon}) \|^2_{L^2(\Omega)} + \lambda \| \Delta\beta_\varepsilon(w_{\lambda,\varepsilon}) \|^2_{L^2(\Omega)} \leqq \| \nabla f \|_{L^2(\Omega)} \cdot \| \nabla\beta_\varepsilon(w_{\lambda,\varepsilon} \|_{L^2(\Omega)}$$

whence

$$\| \nabla\beta_\varepsilon(w_{\lambda,\varepsilon}) \|_{L^2(\Omega)} \leqq C \qquad \text{(constant independent of } \lambda \text{ and } \varepsilon)$$

and then also

$$\lambda \| \Delta\beta_\varepsilon(w_{\lambda,\varepsilon}) \|^2_{L^2(\Omega)} \leqq C,$$

which yields (30).

CONCLUSION

The operator $A : D(A) \to L^1(\Omega)$ generates a non-linear semigroup of contractions $S(t)$, defined by Crandall-Liggett's formula (see [6]):

$$\forall u_0 \in L^1(\Omega), \quad S(t)u_0 = \lim_{n\to\infty} \left(I + \frac{t}{n} A\right)^{-n} u_0 \quad \text{uniformly in } [0, T].$$

Moreover, $u(t) \equiv S(t) u_0 \in C^0([0, T]; L^1(\Omega))$ is the generalized solution in the sense of Crandall-Liggett [6] and Bénilan [1] of the abstract Cauchy problem

$$\frac{du}{dt} + Au = 0 \quad , \quad u(0) = u_0 , \tag{31}$$

or equivalently of problem (P) (see introduction) with $f = 0$.

§ 2. The general case ($f \neq 0$)

Let $f \in L^1(\Omega \times]0, T[)$; let $f_n = f_n^k$ constant in $\left[k \frac{t}{n}, (k+1) \frac{t}{n}\right[$ for $k = 0, \cdots, n-1$ and such that $f_n \to f$ strongly in $L^1(\Omega \times]0, T[)$. Then

$$\forall u_0 \in L^1(\Omega) \quad , \quad U_f(t) u_0 \equiv \lim_{n \to \infty} \prod_{k=1}^{n} \left(I + \frac{t}{n}(A - f_n^k)\right)^{-1} u_0$$

(uniformly in $[0, T]$) is the generalized solution (see [7]) of the abstract Cauchy problem

$$\frac{du}{dt} + Au = f \quad , \quad u(0) = u_0 ,$$

i.e. of problem (P).

Remark. Under natural assumptions on u_0 and g, the solution u of problem (P) with $f = 0$ fulfills a maximum principle: $M_1 \leqq u \leqq M_2$ (M_1, M_2: constants) (by means of an argument similar to one used in [15]). This allows the removal of the assumption on the growth of g (see (2)); therefore the above results apply also to the case of a flux governed by the classical Stefan-Boltzmann radiation law

$$g(\tau) = C(\tau^4 - \tau_0^4) ;$$

here τ denotes the absolute temperature, τ_0 is the temperature of a source and $C > 0$ is a physical constant.

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