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Evolution equations for a class of nonlinear operators

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Analisi matematica. — *Evolution equations for a class of non-linear operators* (*). Nota (**) di ENNIO DE GIORGI (***), MARCO DEGIOVANNI (***), ANTONIO MARINO (****), e MARIO TOSQUES (****), presentata dal Corrisp. E. DE GIORGI.

RIASSUNTO. — Se A è un operatore in uno spazio di Hilbert e V è un sottoinsieme di questo spazio, in molti problemi si è indotti a modificare A sul « bordo » di V in modo da ottenere un operatore $\tilde{A}$ tale che le soluzioni dell'equazione differenziale associata

$$0 \in U' + \tilde{A}(U)$$

non escano da V.

Se V non è convesso, l'operatore $\tilde{A}$ non rientra nei casi classici esaminati, ad esempio, in [1].

In questo lavoro introduciamo alcune classi di operatori che contengono, in qualche caso significativo, quelli del genere sopra considerato e forniamo alcuni teoremi di esistenza e regolarità per le soluzioni dell'equazione differenziale associata.

INTRODUCTION

The classical notion of the evolution equation

$$U' + A(U) = 0 \tag{1}$$

associated to a smooth non-linear operator A has been the object of several studies in order to extend this notion to different situations wherein the non-linear operator is of a more general type. One of the theories developed in this

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direction is that of operators of monotone type, which has made possible the reformulation of the concept of evolution equation in a manner su table for many concrete situations and has furnished many brilliant results. It is, however, important to observe that, already in this theory, the lack of symmetry in the definition of monotone operators leads to a loss of several symmetric properties of the equation associated with them. In fact, if A is a monotone operator in a Hilbert space, a solution U of the equation

$$\text{(2)} \qquad 0 \in U' + A(U)$$

satisfying an assigned initial condition $U(O) = u_0$ (as is well known, such a solution U exists if A is "maximal" monotone) is, in general, defined only in a right (half) neighbourhood of O and is unique only in such a right neighbourhood, as two different solutions may coincide at the same point u_0.

Some results have also been obtained for certain operators close to monotone operators, as for example, for the perturbations of these by Lipschitz operators.

New difficulties arise when it is necessary to work outside this class in a more decisive manner, as happens, for example if, in addition, we also require that the solution of the equation (2), associated to the operator A, should remain in some non convex set V.

Already in the simple case in which V is the closure of a smooth open set Ω in R^n, if A_0 is a monotone operator in R^n and A_1 is a smooth operator then the operator A, defined by

$$A(u) = \begin{cases} \{\alpha + A_1(u) : \alpha \in A_0(u)\} \text{ if } u \in \Omega \\ \{\alpha + A_1(u) - ((\alpha + A_1(u) \mid \nu(u)) \wedge O)\, \nu(u) : \alpha \in A_0(u)\} \\ \qquad \text{if} \quad u \in \partial\Omega \end{cases}$$

where ν is the exterior normal to $\partial\Omega$, is evidently not of the type considered unless Ω is convex.

Along similar lines we can consider the somewhat more complicated case in which A_0 and A_1 are defined on a function space and V is an "irregular" subset of this function space (defined, for example, by a pointwise inequality (cfr. (2,7)) but not by an inequality of integral type): it would naturally be necessary to define the operator in a suitable manner at points of V which, in some sense, constitute the boundary of V.

In this note, as in the papers [3], [4], [5], [6] [7], we shall consider classes of operators which include those to which we have just now referred, in addition, naturally to Lipschitz perturbations of monotone operators.

It is because of these reasons that the solutions of the evolution equations exhibit the same loss of symmetry as we have indicated in the case of monotone operators.

We wish to consider henceforth a class of operators, which seems to us to be particularly significant among those considered.

We shall begin with a definition.

Let Ω be an open subset of a real Hilbert space and $f\colon \Omega \to \mathbf{R} \cup \{+\infty\}$ an extended real valued function on it.

Definition. If $u \in \Omega$ where $f(u) < +\infty$ we shall denote by $\partial^- f(u)$ the set (possibly empty) of all the α in H such that

$$\liminf_{v \to u} \frac{f(v) - f(u) - (\alpha \mid v - u)}{|v - u|} \geq 0$$

we set $\partial^- f(u) = \varnothing$ at other points u of H.

If $\partial^- f(u) \neq \varnothing$, f is said to be sub-differentiable at u and the element of $\partial^- f(u)$ (which, as can easily be verified, is closed and convex) having the minimum norm is denoted by $\operatorname{grad}^- f(u)$.

For many applications it is of interest to consider the operator $\partial^- f$ when f is a lower semi-continuous function and satisfies the following property:

(3) there exists a continuous function $\Phi = \mathbf{R}^3 \to \mathbf{R}$ such that

$$f(v) \geq f(u) + (\alpha \mid v - u) - \Phi(|u|, f(u), |\alpha|)\, |v - u|^2$$

for u, $v \in \Omega$ and $\alpha \in \partial^- f(u)$.

We remark that in the condition (3) above we do not explicitly require that $\partial^- f(u) \neq \varnothing$ for some u.

In the following H will denote a real Hilbert space with $|\cdot|$ and $(.\,|\,.)$ as the norm and the inner product respectively and P (H) will denote the set of all subsets of H.

Any map $A : H \to P(H)$ will be called an operator on H and we set

$$D(A) = \{u \in H \mid Au \neq \varnothing\}$$

$$|A_0 u| = \begin{cases} \inf\{|\alpha| \mid \alpha \in Au\}, & \text{if } u \in D(A) \\ +\infty, & \text{if } u \in H \setminus D(A). \end{cases}$$

If Ω is an open subset of H and if $f : \Omega \to \mathbf{R} \cup \{+\infty\}$ is a function, we set

$$D(f) = \{u \in \Omega \mid f(u) < +\infty\},$$

$$|\nabla f|(u) = \begin{cases} \max\left\{ 0, \limsup\limits_{v \to u} \dfrac{f(u) - f(v)}{|u - v|} \right\} & \text{if } f(u) \leq +\infty \\ +\infty & \text{otherwise} \end{cases}$$

We say that a sequence of functions $(f_h)_{h\in N}$ is equicoercive if for every c in $\mathbf{R}$ the set

$$\bigcup_{h\in N}\{v\in\Omega \mid f_h\leq c\} \quad \text{has a compact closure in } \Omega.$$

Finally, if $U:[t_0, t_0+\delta[\rightarrow H$ is a curve, we set

$$U'_{+}(t_0)=\lim_{t\to t_0^+}\frac{U(t)-U(t_0)}{t-t_0} \quad \text{whenever it exists.}$$

§ 1

Suppose Ω is an open subset of the Hilbert space H, $f:\Omega\rightarrow\mathbf{R}\cup\{+\infty\}$ a lower semi-continuous function and $A:H\rightarrow P(H)$ an operator such that $D(A)\subset D(f)$.

(1.1) *Definition.* If $\omega:\mathbf{R}^6\rightarrow\mathbf{R}$ is a continuous function the operator A is said to be "(ω, f) — monotone" if

$$(\alpha-\beta \mid u-v)\geq-\omega(|u|, |v|, f(u), f(v), |\alpha|, |\beta|)\,|u-v|^2$$

for $u, v\in H$ and $\alpha\in Au$, $\beta\in Av$.

We remark that, if ω is a constant, we get the Lipschitz perturbations of monotone operators (cfr. [1]).

A more general class of examples will be given later (cfr. (2.7)).

(1.2) PROPOSITION. *Let* A *be an* (ω, f)*-monotone operator and let* U, V: $[O, T]\rightarrow H$ *be two absolutely continuous curves in* H *such that*

$$U'(t)\in-AU(t),\ V'(t)\in-AV(t) \qquad \text{a.e. in } [O, T]$$

and

$$\omega(|U(t)|, |V(t)|, f\circ U(t), f\circ V(t), |U'(t)|, |V'(t)|)\in L^1(O, T)$$

(*for instance, this is the case if* U *and* V *are Lipschitz maps and* $f\circ U$, $f\circ V$ *are bounded*).

Then, we have:

$$|U(t)-V(t)|\leq|U(0)-V(0)|\exp\left(\int_0^t\omega(|U(s)|, |V(s)|, f\circ U(s), f\circ V(s), |U'(s)|, |V'(s)|)\,ds\right).$$

Another important condition on A in order to get a solution of the evolution equation (2) is given in the following definition.

(1.3) *Definition.* A is said to be "f-solvable" at a point u of Ω, if for every $c > 0$, there exist $r, \lambda_0 > 0$ such that whenever $0 < \lambda \leq \lambda_0$, $v \in H$ and $\left|\dfrac{u - v}{\lambda}\right| \leq c$, there exists a w in Ω such that $\dfrac{v - w}{\lambda} \in A w$, $\left|\dfrac{v - w}{\lambda}\right| \leq r$ and $f(w) \leq f(u) + r$.

(1.4) PROPOSITION. *Let* A *be an* (ω, f)*-monotone operator which is f-solvable at a point u of* D (A). *Then* A *u is a convex closed subset of* H *and for every* K_0 *there exists* $\delta > 0$, *such that, for every* $K \leq K_0$, *the set*

$$\overline{B(u, \delta)} \cap \{v \in \Omega \mid |A_0 v| \leq K, f(v) \leq K_0\}$$

is closed.

In the following $A_0 u$ will denote the element of Au having the minimum norm.

(1.5) THEOREM. *Let* A *be an* (ω, f)*-monotone operator which is f-solvable at every point of* D (A). *Then, for every* u_0 *in* D (A), *there exists a* $T > 0$ *and a unique Lipschitz curve*

$$U : [O, T[\to H \text{ such that } f \circ U \text{ is bounded and}$$

$$\begin{cases} U'_+(t) = -A_0 U(t), & \forall t \in [O, T[\\ U(0) = u_0. \end{cases}$$

(1.6) THEOREM. *Let the hypotheses of Theorem* (1.5) *hold. If* $(u_n)_n$ *is a sequence in* D (A) *which is convergent to an element u of* D (A) *and sup* $\{|A_0 u_n| \vee f(u_n)\} < +\infty$, *then there exists a* $T > O$ *such that the evolution curves* U_n, U *given by Theorem* (1.5) *with* $U_n(O) = u_n$, $U(O) = u$, *are all defined on* $[O, T]$ *and the sequence* $(U_n)_n$ *converges uniformly to* U *on* $[O, T]$.

(1.7) THEOREM. *Suppose that* A *satisfies the hypotheses of Theorem* (1.5) *and that for every* K_1, K_2 *in* $\mathbf{R}$ *the set* $\{v \in \Omega \mid |A_0 v| \leq K_1, f(v) \leq K_2\}$ *is closed in* Ω.

Let $U : I \to \Omega$ *be a curve in* Ω *such that it is Lipschitz on compact subsets of* I, $f \circ U$ *is bounded on compact subsets of* I *and* $U'(t) \in -A U(t)$ *a.e. on* I.

Then we have the following:

a) $U(t) \in D(A)$ for every t in I;

b) $U'_+(t) = -A_0 U(t)$ if $t \in I$, $t < \sup I$;

c) the map $t \to A_0 U(t)$ is continuous on I outside possibly a countable subset of I;

d) if I is the maximal interval of existence of U and $\sup I < +\infty$ then

$$\lim_{t \to \sup I} |A_0 U(t)| \vee f \circ U(t) \vee d(U(t), \partial\Omega)^{-1} = +\infty$$

(1.8) THEOREM. *Suppose that*

a) A *is an* (ω, f)*-monotone operator with*

$\omega = \chi(x_1, x_2, x_3, x_4)(1 + x_5^2 + x_6^2)$ *where* $\chi : \mathbf{R}^4 \to \mathbf{R}$

is a continuous function,

b) A *is f-solvable at every point of* D(A) *and for every* K_1, K_2 *in* $\mathbf{R}$, *the set* $\{v \in \Omega \mid |A_0 v| \leq K_1, f(v) \leq K_2\}$ *it closed in* Ω,

c) *there exists a continuous function* $\theta : \mathbf{R}^3 \to \mathbf{R}$ *such that*

$$f(v) \geq f(u) - \theta(|u|, f(u), |\operatorname{grad}^- f(u)|)\, |v - u|$$

if $v \in H$, *and* $u \in D(\partial^- f)$,

d) $D(A) \subset D(\partial^- f)$ *and there exist* $\varepsilon > 0$, *two continuous functions* $\psi_1 : \mathbf{R} \to \mathbf{R}$, $\psi_2 : \mathbf{R}^2 \to \mathbf{R}$ *such that*:

$$(\operatorname{grad}^- f(u) \mid A_0 u) \geq \varepsilon |A_0 u|^2 - \psi_1(f(u)) \quad \text{if } u \in D(A)$$
$$|\operatorname{grad}^- f(u)| \leq \psi_2(|A_0 u|, f(u)) \qquad \text{if } u \in D(A).$$

Then, for every u_0 in $F = \bigcup_{K \in R} \overline{\{v \in D(A) \mid f(v) \leq K\}}$,
there exist a $T > 0$ and a unique curve U in $H^{1,2}(0, T; H)$ such that $f \circ U$ is bounded and

$$\begin{cases} U'_+(t) = -A_0 U(t) & \text{for every } t \text{ in } [0, T[\\ U(0) = u_0 . \end{cases}$$

Note that, even if A is a maximal monotone operator (cfr. [1]), then there does not, in general, exist a function f satisfying the hypothesis of Theorem (1.8) and such that $D(A) \subsetneq F$.

§ 2

(2.1) THEOREM. *Suppose* $\partial^- f$ *to be an* (ω, f)*-monotone operator. Then* $\partial^- f$ *is f-solvable at every point u of* Ω *such that* $\partial^- f(u) \neq \varnothing$.

Moreover if $u \in \Omega$,

$$|\nabla f|(u) < +\infty \quad \text{if and only if } \partial^- f(u) \neq \varnothing$$

and

$$|\nabla f|(u) = |\operatorname{grad}^- f(u)|, \text{ if } \partial^- f(u) \neq \varnothing .$$

(2.2) THEOREM. *Let us suppose* $\partial^- f$ *to be an* (ω, f)*-monotone operator. Then for every* u_0 *such that* $\partial^- f(u_0) \neq \varnothing$, *there exists a* $T > O$ *and a unique Lipschitz curve* $U : [O, T[\to H$, *such that* $f \circ U$ *is Lipschitz continuous and*

$$\begin{cases} U'_+(t) = -\operatorname{grad}^- f(U(t)) & \text{if } t \in [O, T[\\ (f \circ U)'_+(t) = -|\operatorname{grad}^- f(U(t))|^2 & \text{if } t \in [O, T[\\ U(O) = u_0 . \end{cases}$$

The class of functions which satisfy the hypotheses of Theorem (2.2), represents a generalization of the class of (p, q)-convex functions introduced in [4], whose definition we recall here:

(2.3) *Definition.* Let p, q be two real numbers with $p \geq O$. Then a lower semicontinuous function $f : H \to \mathbf{R} \cup \{+\infty\}$ is said to be (p, q)-convex if for every pair u, v in H, there exists a z in H such that

$$\left| z - \frac{u+v}{2} \right| \leq p\,|u - v|^2 , \quad 2f(z) \leq f(u) + f(v) - 2q\,|u - v|^2 .$$

It is easy to deduce, from the results of [4], that if f is (p, q)-convex, the hypotheses of Theorem (2.2) are satisfied with $\Omega = H$, $\omega(x_1, \cdots, x_6) = 16(p(x_5 + x_6) + q^-)$.

(2.4) *Remark.* A class of functions, satisfying the hypotheses of Theorem (2.2), which seems to be useful for applications, consists of functions f such that

$$(2.5) \quad \begin{cases} \text{there exists a continuous function } \Phi : \mathbf{R}^3 \to \mathbf{R} \text{ such that} \\ f(v) \geq f(u) + (\alpha \mid v - u) - \Phi(|u|, f(u), |\alpha|)\,|v - u|^2 \\ \text{i } u, v \in \Omega, \ \alpha \in \partial^- f(u) . \end{cases}$$

The class of functions introduced above, possesses a certain stability property described in the following theorem.

(2.6) THEOREM. *Let* $(f_h)_{h \in N}$ *be a sequence of equi-coercive, functions satisfying* (2.5), *with a* Φ *independent of* h.
Let us suppose the existence of

$$\Gamma(H^-) \lim_h f_h = f : \Omega \to \mathbf{R} \cup \{+\infty\} \qquad \text{(cfr. [2])} .$$

Then f *also satisfies* (2.5) *with the same* Φ.

We can easily deduce, from the results of [5], the following remark.

(2.7) *Remark.* Let $h : \mathbf{R}^n \to \mathbf{R}$ be a C^3 function such that $h^{-1}(O)$ is a non-empty compact set such that grad $h(x) \neq O$ where $h(x) = O$.

Let P, Q be in $h^{-1}(]-\infty, O])$. If we set $\Omega = H = (L^2(O, 1))^n$, and f is the function defined by

$$f(u) = \begin{cases} 1/2\displaystyle\int_0^1 |u'(t)|^2 ds & \text{if } u \in (H^{1,2}(O,1))^n,\ u(0) = P,\ u(1) = Q, \\ & \quad h(u(t)) \leq O,\ \forall t \in [O,1] \\ +\infty & \text{elsewhere,} \end{cases}$$

then f satisfies the condition (2.5) with

$$\Phi(x_1, x_2, x_3) = \rho(x_2)(1 + x_3^2)$$

where $\rho : \mathbf{R} \to \mathbf{R}$ is a continuous function.

Moreover, in this case,

$$\partial^- f(u) \neq \varnothing \quad \text{if and only if } f(u) < +\infty,\ u \in (H^{2,2}(O,1))^n$$

and $\alpha \in \partial^- f(u)$ if and only if α is of the form

$\alpha = -u'' + \beta$ grad $h(u)$ where $\beta \in L^2(O, 1)$, $\beta \geq O$ and

$\beta(s) = O$ if $h(u(s)) < O$.

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