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Weakly hyperbolic equations of second order well-posed in some Gevrey classes

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Equazioni a derivate parziali. — *Weakly hyperbolic equations of second order well-posed in some Gevrey classes.* Nota (*) di ENRICO JANNELLI (**), presentata dal Corrisp. E. DE GIORGI.

RIASSUNTO. — L'equazione $u_{tt} = \sum_{ij=1}^{n} (a_{ij}(x, t)\, u_{x_j})_{x_i}$ in condizioni di debole iperbolicità $\left(\sum_{ij=1}^{n} a_{ij}(x, t)\, \xi_i\, \xi_j \geq 0\right)$, è b n posta negli spazi di Gevrey $\gamma_{loc}^{(s)}$ con $1 \leq s < 1 + \frac{\sigma}{2}$, purché a_{ij} sia di Gevrey in x di ordine s e risulti

$$\left[\sum_{ij=1}^{n} a_{ij}(x, t)\, \xi_i\, \xi_j\right]^{1/\sigma} \in \mathrm{BV}([0, T]\,;\, L_{loc}^{\infty})$$

1. INTRODUCTION

In this work we shall deal with the following equation:

$$(1) \qquad \begin{cases} u_{tt} = \sum_{ij=1}^{n} (a_{ij}(x, t)\, u_{x_j})_{x_i} & \text{on } \mathbf{R}_x^n \times [0, T] \\ u(x, 0) = \varphi(x) \\ u_t(x, 0) = \psi(x). \end{cases}$$

The matrix a_{ij} is a real symmetric matrix having the following properties:

i) $\sum_{ij=1}^{n} a_{ij}(x, t)\, \xi_i\, \xi_j \geq 0 \qquad \forall t \in [0, T] \;,\; \forall (x, \xi) \in \mathbf{R}^{2n}$;

ii) There exists a number $\sigma \geq 1$ such that, $\forall$K compact subset of $\mathbf{R}_x^n$, $\forall \xi \in \mathrm{R}^n$

$$\left[\sum_{ij=1}^{n} a_{ij}(x, t)\, \xi_i\, \xi_j\right]^{1/\sigma} \in \mathrm{B\,V}([0, T];\, L^{\infty}(K));$$

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moreover

$$\sup_{|\xi|=1} \left\| \left[\sum_{ij=1}^{n} a_{ij}(x, t)\, \xi_i\, \xi_j \right]^{1/\sigma} \right\|_{BV((0,T);\, L^{\infty}(K))} = M_K < +\infty \ ;$$

iii) For any K compact subset of $\mathbf{R}_x^n$ there exist some positive constants Λ_K, A_K such that

$$| D_x^{\alpha} a_{ij}(x, t) | \leq \Lambda_K A_K^{|\alpha|} | \alpha |^{s|\alpha|} \ \forall (x, t) \in K \times [0, T], \forall \alpha \in \mathbf{N}^n$$

for a fixed number $s \geq 1$ (i.e. the matrix a_{ij} belongs to the Gevrey class $\gamma_{loc}^{(s)}$ in x, uniformly with respect to t).

From these hypotheses we have obtained the following

THEOREM 1. *Let* $\varphi(x)$, $\psi(x) \in \gamma_{loc}^{(s)}$. *Then problem* (1) *has one and only one solution* $u \in C^1([0, T]; \gamma_{loc}^{(s)})$ *provided that*

$$1 \leq s < 1 + \frac{\sigma}{2}$$

Remark 1. If $a_{ij}(x, t) = a_{ij}(t)$, the result of Theorem 1 is contained in [2], where a class of counter-examples shows that this result, in a certain sense, cannot be improved.

In connection with Theorem 1, for coefficients hölder continuous in t, see also [4].

2. Sketch of the proof

The idea of the proof is to approximate problem (1) by means of a family of strictly hyperbolic problems with sufficiently smooth coefficients and to show that the corresponding solutions are bounded in $C^1([0, T]; \gamma_{loc}^{(s)})$, in order to obtain a sequence converging in $C^1([0, T]; \gamma_{loc}^{(s)})$ to a solution $u(x, t)$ of (1). After this, the uniqueness of the solution of (1) is obtained by a duality method.

In order to illustrate the situation, let us consider the simplest case of (1), i.e. the equation

$$(2) \qquad \begin{cases} u_{tt} = a(t)\, u_{xx} & \text{on } \mathbf{R}_x \times [0, T] \\ u(x, 0) = \varphi(x) \\ u_t(x, 0) = \psi(x) \end{cases}$$

where $a(t) \geq 0$ and $a(t)^{1/\sigma} \in \mathrm{BV}([0, T])$; moreover, as a further simplification, we suppose $a(t) \in C^1([0, T])$ (this last hypothesis is removable).

From [3] it is known that problem (1) is well-posed in $\gamma_{loc}^{(1)}$ (the space of the real analytic functions) without any assumption of regularity in t as regards the coefficients a_{ij}; therefore, we can suppose that $s > 1$ and that, according to the finite speed of propagation, $\varphi(x)$ and $\psi(x)$ have compact support, i.e. they belong to $\gamma_0^{(s)}$.

Equations of type (2) are studied in [2] by means of the Fourier-Laplace transform; now, let us see how we can obtain the result of Theorem 1 for the equation (2) using our method of approximation by strictly hyperbolic equations.

Let $h \in \mathbf{N}$. Define:

$$a_h(t) = a(t) + h^{-\sigma};$$

$$(3) \qquad \begin{cases} (u_m)_{tt} = a_m(t)(u_m)_{xx} & \text{on } \mathbf{R}_x \times [0, T] \\ (u_m)(x, 0) = \varphi(x) \\ (u_m)_t(x, 0) = \psi(x) \end{cases}$$

$$E_{h,m}(t) = a_h(t) \int_{\mathbf{R}} [D_x^h u_m(x, t)]^2 \, dx + \int_{\mathbf{R}} [D_x^{h-1} D_t u_m(x, t)]^2 \, dx$$

($E_{h,m}(t)$ is a sort of approximated energy of $D_x^{h-1} u_m$).

From the fact that $D_x^k u_m$ is a solution of (3) $\forall k \in \mathbf{N}$ it follows easily that

$$(4) \qquad E'_{h,m}(t) \leq \frac{|a'_h(t)|}{a_h(t)} E_{h,m}(t) + h^{-\sigma/2} [E_{h,m}(t) + E_{h+1,m}(t)]$$

if $h \leq m - 1$, while

$$(5) \qquad E'_{m,m}(t) \leq \frac{|a'_m(t)|}{a_m(t)} E_{m,m}(t).$$

Using Gronwall's lemma and the inequalities (4) (iterated $m - 1$ times) and (5) we obtain

$$(6) \qquad E_{1,m}(t) \leq \sum_{1}^{m}{}_h \lambda_h(t) E_{h,m}(0)$$

where

$$\lambda_h(t) = \exp\left\{\int_0^t \frac{|a'_h(s)|}{a_h(s)} \, ds\right\} \cdot t^{h-1} e^t [(h-1)!]^{-(1+\sigma/2)}.$$

From the fact that $a(t)^{1/\sigma} \in \mathrm{BV}([0, T])$ it follows

$$\lambda_h(t) \leq e^{Bt} (te^{Bt})^{h-1} \cdot [(h-1)!]^{-(1+\sigma/2)} \tag{7}$$

where B is a positive constant depending only on $\| a(t)^{1/\sigma} \|_{\mathrm{BV}((0,T))}$ while, being $\varphi(x)$, $\psi(x) \in \gamma_0^{(s)}$, we can estimate

$$E_{h,m}(0) \leq CA^{h-1} [(h-1)!]^s. \tag{8}$$

Substituting (7) and (8) in (6) we get

$$E_{1,m}(t) \leq Ce^{Bt} \sum_0^{m-1}{}_h \frac{(Ate^{Bt})^h}{(h!)^{1+\sigma/2-s}} \leq Ce^{Bt} \sum_0^{\infty}{}_h \frac{(Ate^{Bt})^h}{(h!)^{1+\sigma/2-s}} \tag{9}$$

and the last series converges to a number independent of m, this convergence being guaranteed by the fact that $s < 1 + \sigma/2$.

Analogously, one can prove other estimates on the $E_{h,m}(t)$, independent of m, of the type

$$E_{h,m}(t) \leq C (2A)^h e^{Bht} (h!)^s \sum_0^{\infty}{}_h \frac{(2Ate^{Bt})^h}{(h!)^{1+\sigma/2-s}} . \tag{10}$$

By means of (9) and (10) we get that the sequence $u_m(x, t)$ is bounded in $C^1([0, T]; \gamma_0^{(s)})$; therefore, there exists a subsequence u_m that converges to a function $u(x, t) \in C^1([0, T]; \gamma_0^{(s)})$ which is a solution of (2).

This method of approximated energies in L^2-norm of the solution and its derivatives, unlike the Fourier-Laplace transform, also works very well in the case in which the coefficients a_{ij} depend on x; clearly, computations are much more complicated by the fact that the successive derivatives $D_x^\alpha u$ of the solution u don't solve the original equation (1), but a modified equation with the same principal part of (1) plus other terms depending on the derivatives of the coefficients and of the solution up to the order $|\alpha| - 1$.

3. A Theorem for strictly hyperbolic systems

Using just the same techniques, we are also able to prove the following theorem, regarding strictly hyperbolic systems:

THEOREM 2. *Let us consider the system*

$$\begin{cases} U_t = \sum_1^n {}_h A_h(x, t) U_{x_h} + B(x, t) U & \text{on } \mathbf{R}_x^n \times [0, T] \\ U(x, 0) = \varphi(x) \end{cases} \tag{11}$$

where A_h, B *are* N×N *matrices and* φ *is an* N-*vector.*

We suppose that:

i) problem (11) is strictly hyperbolic, i.e. the equation

$$\det\left(\lambda I - \sum_{1}^{n}{}_h A_h (x , t) \xi_h\right) = 0$$

has N real and distinct roots $\lambda = \lambda (x , t ; \xi)$;

ii) $A_h (x , t) \in C^{0,\alpha} ([0 , T]; \gamma^{(s)}_{loc})$

$B (x , t) \in L^1 ([0 , T] ; \gamma^{(s)}_{loc})$

(roughly speaking, the matrices A_h are hölder-continuous in t and Gevrey in x).

Then, for any vector $\varphi (x) \in \gamma^{(s)}_{loc}$ problem (11) has one and only one solution $U \in C ([0 , T] ; \gamma^{(s)}_{loc})$ provided that

$$1 \leq s < \frac{1}{1 - \alpha}.$$

This result, for a scalar operator of order 2, has been proved by T. Nishitani in [4], using quite different techniques; however, the first result in this direction, regarding second order hyperbolic equations with time dependent coefficients, is due to F. Colombini, E. De Giorgi and S. Spagnolo (see [1]).

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