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# Rendiconti

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## Uniqueness theorems for steady, compressible, heat-conducting fluids: bounded domains

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**Fisica matematica.** — *Uniqueness theorems for steady, compressible, heat-conducting fluids: bounded domains* (*). Nota I di MARIAROSARIA PADULA (**), presentata (***) dal Socio D. GRAFFI.

RIASSUNTO. — Si fornisce un teorema di unicità per moti stazionari regolari di fluidi compressibili, viscosi, termicamente conduttori, svolgentisi in regioni limitate dello spazio fisico.

§ 1. During the last twenty years, the well-posedness theory for viscous compressible fluids has received remarkable contributions from the early papers of D. Graffi and J. Serrin on uniqueness [1, 2, 3] to the more recent works of A. Matsumura and T. Nishida [4, 5] on global (in time) existence (cf., also, [8] and the literature cited therein). All of these results concern the equations governing the unsteady motions of thermally conducting fluids. Moreover, the present writer has recently proved existence and uniqueness theorems for steady motions of barotropic, viscous fluids in bounded domains [10]. This simplified scheme has been adopted in order not to obscure the underlying ideas and to present the technical tools as simply as possible. Obviously, the heat-conducting model may represent a more realistic and more interesting situation from the physical point of view (e.g., the Benard problem without Boussinesq approximation [6]).

In this note, employing and generalizing the methods introduced in [7], we give a contribution to solving the above problem for thermally conducting fluids. Precisely, we prove a uniqueness theorem for smooth solutions to the equations governing the motion of a viscous, compressible, ideal and polytropic fluid occurring in a bounded domain $\Omega$, whose boundary is rigid and of infinite thermal conductivity (velocity and temperature ascribed). The smoothness assumptions on solutions are the usual ones [1, 2] and include the existence of a strictly positive lower bound for both temperature and density. We notice, further, that our constitutive hypotheses are by no means restrictive and that they are made here only in order to simplify the proof of the theorem and to state the sufficient conditions for uniqueness in terms of well-suited nondimensional parameters.

The plane of the work is the following. In section 2, after recalling a preliminary lemma, we write the equations of motion in a suitable dimensionless form and define the regularity class $\mathscr{S}$ where the uniqueness is proved. Moreover, we state the uniqueness result and clarify the "smallness" assumptions we

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make on Mach, Prandtl and Reynolds numbers and on the ratio of the maximum of $\nabla\rho$ and the minimum of the absolute temperature associated to the motions. It is easy to convince oneself that a more detailed analysis would allow us to deduce the well known uniqueness results for classical incompressible flows [9] in the limit of " vanishing compressibility " [14]. Finally, in section 3 we prove the uniqueness theorem.

We end by noticing that, though the class $\mathscr{I}$ is non empty, as shown in section 2, it would be desirable to provide existence in it and this will be the object of future research.

§ 2. We commence this section by recalling a result which is of basic importance in this note and whose proof is given in [13] [(1)].

LEMMA 1. *Let* $\Omega \in C^1$. *For any* $\phi \in \mathbf{L}^2(\Omega)$ *such that* $\int_\Omega \phi \, dx = 0$, *there exists at least one function* $\varphi \in \mathbf{W}_0^{1,2}$ (i.e., $\partial_i \varphi \in \mathbf{L}^2(\Omega)$, $\varphi = 0$ *on* $\partial\Omega$, $\partial_i = \partial/\partial x_i$, $(x_i)$ *rectangular coordinates) such that*

$$
\left\{
\begin{array}{ll}
\nabla \cdot \varphi = \phi & \text{in } \Omega , \\
|\nabla\varphi|_2 \leq c\,|\phi|_2
\end{array}
\right.
\tag{1}
$$

*where* $|\cdot|_2$ *denotes the usual* $L^2$*-norm and c denotes a positive constant depending only on the regularity of* $\partial\Omega$.

As we mentioned in section 1, we shall be concerned with polytropic fluids, i.e., fluids for which the internal specific energy $\bar{\varepsilon}$ is proportional to the absolute temperature $\bar{\theta}$, the proportionality constant $c_V$ being the specific heat at constant volume [11]. Moreover, the ideal gas assumption implies $\bar{p} = R^* \bar{\rho}\,\bar{\theta}$, where $\bar{p}$ is the pressure, $\bar{\rho}$ the density and $R^*$ the gas constant. However, as we have already said, the more general case $\bar{\varepsilon} = \bar{\varepsilon}(\bar{\rho}, \bar{\theta})$, $\bar{p} = \bar{p}(\bar{\rho}, \bar{\theta})$ presents no conceptual difficulty and can be handled by the same methods, provided the constitutive relations are sufficiently smooth functions of their arguments (cf. [1] pp. 103–104).

As is well known, for a polytropic ideal fluid the nondimensional analysis is allowed [15]. In particular, we have the following dimensionless equations governing the steady motions of a compressible, viscous, polytropic ideal gas

$$
\left\{
\begin{array}{l}
R\rho\boldsymbol{v} \cdot \nabla\boldsymbol{v} - \Delta_2 \boldsymbol{v} - (\vartheta - 1)\nabla\nabla \cdot \boldsymbol{v} = -(R/M^2)\nabla p + R\rho\boldsymbol{f} \\
(c_V/c_p)\,Pr\,R\rho\,\boldsymbol{v}\cdot\nabla\theta - \Delta_2\theta = -(R^*/c_p)\,Pr\,Rp\,\nabla\cdot\boldsymbol{v} + Pr\,[(\vartheta - 1)(\nabla\cdot\boldsymbol{v})^2 + \\
\qquad + 2\mathbf{D}:\mathbf{D}] \\
\nabla \cdot (\rho\boldsymbol{v}) = 0 ,
\end{array}
\right.
\tag{2}
$$

(1) For a given function space V, we set $\mathbf{V} = [V]^3$.

with $p = \rho\theta$ and $\mathbf{D} = (\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^{T})/2$. In (2) $\rho$, $\boldsymbol{v}$, $\theta$ represent the (dimensionless) density, velocity and temperature, respectively, while $\boldsymbol{f}$ is the (dimensionless) force per unit mass. Moreover, denoting by $\rho^*$, $\boldsymbol{v}^*$, $\theta^*$ comparison dimensional quantities, for the above fields and by $d$ a comparison length, we set

$$M^2 \equiv (v^*)^2/R^* \theta^* \quad ; \quad R \equiv d\rho^* \mid v^* \mid/\mu \; ;$$
$$Pr \equiv c_p \, \mu/\chi \quad ; \quad \vartheta = (\lambda + 2\mu)/\mu \, ,$$

where $\mu$, $\lambda$ are the Lamè coefficients and $\chi$ is the thermal conductivity. The nondimensional numbers $M$, $R$, $Pr$, $\vartheta$ are usually called Mach, Reynolds, Prandtl and viscosity numbers, respectively [12]. Moreover, the Clausius–Duhem inequality implies [12]: $\vartheta \geq 4/3$.

To system (2) we append the boundary conditions

$$\left\{ \begin{array}{l} \boldsymbol{v} \mid_{\partial\Omega} = \boldsymbol{v}_* \\ \theta \mid_{\partial\Omega} = \theta_* \, , \end{array} \right. \tag{3}$$

here $\boldsymbol{v}$, $\theta_*$ are regular functions on $\partial\Omega$.

Together with (3) we shall suppose that the total mass is ascribed, i.e.,

$$0 < \int_\Omega \rho \, dx = \mathscr{M} < \infty \, . \tag{4}$$

In the sequel, problem (2)–(3)–(4) will be denote by $\mathscr{B}$. We shall consider, for $\mathscr{B}$, only classical solutions in the following sense

$$\mathscr{I} = \{ (\rho \, , \boldsymbol{v} \, , \theta) \in C^1_p(\overline{\Omega}) : m_\rho \leq \rho \; ; m_\theta \leq \theta \; ; \max \{ \mid \nabla \cdot \boldsymbol{v} \mid , \mid \nabla\rho \mid \} \leq k_1 \; ;$$
$$\max \{ \mid \nabla \cdot \boldsymbol{v} \mid/\rho \, , \rho \, , \boldsymbol{v} \, , \theta \, , \mid \nabla\theta \mid , \mid \nabla \boldsymbol{v} \mid \} \leq k \} \, .$$

Here, $C^1_p(\overline{\Omega})$ denotes the space of functions having first derivatives piecewise continuous and bounded. In this way, we may take into account discontinuity waves but not shock waves. This latter fact has an exact analogy with the non-stationary case [1] in which the same assumption is made on solutions for which uniqueness is proved.

We put

$$a_1 = 8 \, [\sqrt{3} \, (\vartheta - 1) + c \, (1 + 2 R k^2 \nu)]/7 \; ;$$
$$a_2 = 2^7 \, M^2 \, k \, [3 \, (\vartheta - 1)^2 + c^2 \, (1 + 2 R k^2 \nu)^2]/49 \, R m_\theta \; ;$$
$$a_3 = \nu \, k \, [M^{-2} \, (\sqrt{3} + 8 \, k k_1 \, \nu/7 \, m_\theta)] + 8 \, \gamma/7 \; ;$$
$$a_4 = \{ 2^7 \, k^3 \, k_1 \, \nu^2/49 \, M^2 \, m_\theta^2 \} \; ;$$
$$b_1 = (R^* \, k^2 \, \nu^2/c_p) + 8 \nu \, (c_V + R^* \, \nu) \, k^3/7 c_p \, m_\theta \; ;$$
$$b_2 = k \, (3 \, \vartheta - 1) \; ;$$
$$b_3 = R k^2 \, \nu \, (c_V + \sqrt{3} \, R^*)/c_p + 2 k \nu \, [\sqrt{3} \, (\vartheta - 1) + 2] +$$
$$+ \, 8 \, M^2 k^2 \, (c_V + R^* \, \nu) \, [\sqrt{3}(\vartheta + 1) + c \, (1 + 2 R k^2 \, \nu)]/7 \, c_p \, m_\theta ;$$
$$\gamma = (b\nu + \sup \mid \boldsymbol{v}^2 \mid)/m_\theta \, ,$$

where, $b = \max_{\Omega} |\boldsymbol{f}|$ and $\nu = \max_{W_0^{1,2}} \int_{\Omega} w^2 \, dx \Big/ \int_{\Omega} (\nabla w)^2 \, dx$. We are, now, in position to state the main theorem.

UNIQUENESS THEOREM. – *Assume* $\Omega \in C^1$, $f \in C^0(\overline{\Omega})$ *and let numbers* $a_1, \cdots, a_4, b_1, \cdots, b_3, \gamma, k_1, R, M, Pr$ *verify the following relations*

$$
\left\{
\begin{aligned}
&(\gamma\, M^2/m_\theta) < \min\{1/8\,c\,,\,1/6\,a_1\}\,,\\
&R < \min\{1/6\,k^2\,\nu\,,\,1/[a_3 + 8\,(2\,a_4 + a_3)\,(2\,b_2 + b_3)]\}\,,\\
&Pr < \min\{1/2R\,b_1\,,\,1/8\,b_3\}\,,\\
&(k_1/m_\theta) < 1/6\,(a_2 + a_1)\,.
\end{aligned}
\right.
\tag{5}
$$

*Then, there exists at most one solution* $(\rho, \boldsymbol{v}, \theta) \in \mathscr{I}$ *to problem* $\mathscr{B}$.

*Remark 1.* The regularity class $\mathscr{I}$ where uniqueness is proved can be fairly enlarged. In fact, on the one hand, the local bounds on the velocity field (i.e., $|\boldsymbol{v}|, |\nabla \boldsymbol{v}| < k$) can be replaced by the weaker assumption $\int_{\Omega} |\nabla \boldsymbol{v}|^2 \, dx < k$; on the other hand, employing the " energy identity ", by direct calculation one shows that $\int_{\Omega} |\nabla \boldsymbol{v}|^2 \, dx$ is estimated in terms of the $L^2$-norm of $\rho \boldsymbol{f}$ and of $\rho\theta$. Thus, the assumption on the velocity is a *consequence* of those made on $\rho$ and on $\theta$.

*Remark 2.* It is easy to verify that (5) are self-consistent.

*Remark 3.* It is not difficult to find, in the limit of vanishing compressibility ($M \to 0$, $k_1 \to 0$), the usual relations ensuring uniqueness for incompressible viscous, steady motions [9, 14].

3. Let ($\hat{\rho} = \rho + \rho'$, $\hat{\boldsymbol{v}} = \boldsymbol{v} + \boldsymbol{u}$, $\hat{\theta} = \theta + \theta'$) and $(\rho, \boldsymbol{v}, \theta)$ be two solutions to the boundary value problem $\mathscr{B}$ in the class $\mathscr{I}$. Their difference $(\rho', \boldsymbol{u}, \theta')$ will, then, verify the following system

$$
\left\{
\begin{aligned}
&R\hat{\rho}\hat{\boldsymbol{v}}\cdot\nabla\boldsymbol{u} + R[\rho\boldsymbol{u} + \rho'(\boldsymbol{v}+\boldsymbol{u})]\cdot\nabla\boldsymbol{v} - \Delta\boldsymbol{u} - (\vartheta - 1)\nabla\nabla\cdot\boldsymbol{u} = RM^{-2}\nabla p' + R\rho'\boldsymbol{f}\\
&RPr(c_V/c_p)\{\hat{\rho}\hat{\boldsymbol{v}}\cdot\nabla\theta' + [\rho\boldsymbol{u} + \rho'\hat{\boldsymbol{v}}]\cdot\nabla\theta\}\,\Delta_2\theta' = -RPr(R^*/c_p)(\hat{p}\nabla\cdot\boldsymbol{u} + p'\nabla\cdot\boldsymbol{v})\\
&\qquad + Pr\,(\vartheta + 1)\,[(\nabla\cdot\boldsymbol{u})^2 + 2\nabla\cdot\boldsymbol{v}\nabla\cdot\boldsymbol{u}] + 2\,Pr\,[\mathbf{D}' : \mathbf{D}' + 2\mathbf{D} : \mathbf{D}']\\
&\nabla\cdot[\rho\boldsymbol{u} + \rho'\,\hat{\boldsymbol{v}}] = 0\\
&\boldsymbol{u}\,|_{\partial\Omega} = \theta'\,|_{\partial\Omega} = 0\\
&\int_{\Omega} \rho'\,dx = 0
\end{aligned}
\right.
\tag{6}
$$

where $p' = \theta\rho' + \hat{\rho}\theta'$ and $\mathbf{D}' = [\nabla\boldsymbol{u} : \nabla\boldsymbol{u}^T]/2$.

Let us multiply $(6)_{1,2}$ by $\boldsymbol{u}$ and $\theta'$, respectively, and integrate over $\Omega$ to obtain

$$(7)\qquad \begin{cases} |\Delta \boldsymbol{u}|_2^2 + (\vartheta - 1)\,|\nabla\cdot\boldsymbol{u}|_2^2 = (\mathrm{R}/\mathrm{M}^2)\displaystyle\int_\Omega \theta\rho'\,\nabla\cdot\boldsymbol{u}\,\mathrm{d}x + \mathrm{F}_1 \\ |\Delta\theta'|_2^2 = \mathrm{F}_2\,, \end{cases}$$

where use has been made of $(6)_{3,4,5}$. We set, also,

$$\begin{cases} \mathrm{F}_1 = \dfrac{\mathrm{R}}{\mathrm{M}^2}\displaystyle\int_\Omega \hat\rho\,\theta'\,\nabla\cdot\boldsymbol{u}\,\mathrm{d}x + \mathrm{R}\int_\Omega [\rho'\boldsymbol{f}\cdot\boldsymbol{u} + \hat\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{v} + \rho'\,\boldsymbol{v}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{v}]\,\mathrm{d}x \\ \mathrm{F}_2 = \dfrac{\mathrm{R}\Pr}{c_p/c_{\mathrm{V}}}\displaystyle\int_\Omega (\hat\rho\boldsymbol{u} + \rho'\,\boldsymbol{v})\cdot\nabla\theta'\,\theta\,\mathrm{d}x - \dfrac{\mathrm{R}\Pr}{c_p/\mathrm{R}^*}\int_\Omega (\hat p\nabla\cdot\boldsymbol{u} + p'\nabla\cdot\boldsymbol{v})\,\theta'\,\mathrm{d}x + \\ \qquad + \Pr\displaystyle\int_\Omega [(\vartheta - 1)(\nabla\cdot\boldsymbol{u})^2 + 2\mathbf{D}' : \mathbf{D}']\,\theta'\,\mathrm{d}x + \\ \qquad + 2\Pr\displaystyle\int_\Omega [(\vartheta - 1)\,\nabla\cdot\boldsymbol{v}\nabla\cdot\boldsymbol{u} + 2\mathbf{D} : \mathbf{D}']\,\theta'\,\mathrm{d}x\,. \end{cases}$$

To deduce an equation for $\rho'$, let us pick on the left hand side of $(6)_1$ the term $-(\mathrm{R}/\mathrm{M}^2)\,\nabla(\theta\rho')$ and multiply this equation by $\boldsymbol{\varphi}\in \mathbf{W}_0^{1,2}$, it results

$$(8)\qquad (\mathrm{R}/\mathrm{M}^2)\int_\Omega \theta\rho'\,\nabla\cdot\boldsymbol{\varphi}\,\mathrm{d}x = -(\mathrm{R}/\mathrm{M}^2)\int_\Omega \hat\rho\theta'\nabla\cdot\boldsymbol{\varphi}\,\mathrm{d}x - \mathrm{R}\int_\Omega \rho'\boldsymbol{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x +$$
$$+ \int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{\varphi}\,\mathrm{d}x + (\vartheta - 1)\int_\Omega \nabla\cdot\boldsymbol{u}\,\nabla\cdot\boldsymbol{\varphi}\,\mathrm{d}x - \mathrm{R}\int_\Omega \hat\rho\,\hat{\boldsymbol{v}}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}x +$$
$$+ \mathrm{R}\int_\Omega (\hat\rho\boldsymbol{u} + \rho'\,\boldsymbol{v})\cdot\nabla\boldsymbol{\varphi}\cdot\boldsymbol{v}\,\mathrm{d}x\,.$$

Now, we let in (8) $\nabla\cdot\boldsymbol{\varphi}$ varying in $\mathrm{L}^2(\Omega)$ and $|\nabla\cdot\boldsymbol{\varphi}|_2 = 1$; by lemma 1 we deduce

$$(9)\qquad (\mathrm{R}/\mathrm{M}^2)\,|\theta\rho'|_2 \le (\mathrm{R}/\mathrm{M}^2)\,|\hat\rho\theta'|_2 - \mathrm{R}\int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}\,\rho'\,\mathrm{d}x + (\vartheta - 1)\,|\nabla\cdot\boldsymbol{u}|_2 +$$
$$+ c\,|\nabla\boldsymbol{u}|_2 + \mathrm{R}c\,|\hat\rho\hat{\boldsymbol{v}}\boldsymbol{u}|_2 + \mathrm{R}c\,\|\boldsymbol{v}|\,(\hat\rho\,|\boldsymbol{u}| + \rho'\,|\boldsymbol{v}|)\,|_2\,.$$

To obtain sufficient conditions for uniqueness to hold we employ, now, properties of solutions in the class $\mathscr{S}$. From (9), by using Schwarz and Poincarè inequalities, we thus have

$$(\mathrm{R}/\mathrm{M}^2)\,[m_\theta - \mathrm{M}^2 c\,(b\nu + k^2)]\,|\,\rho'\,|_2 \leq \frac{\mathrm{R}k}{\mathrm{M}^2/\nu}\,|\,\nabla\theta'\,|_2 + $$
$$+ c\,(1 + 2\,\mathrm{R}k^2\nu)\,|\,\nabla \boldsymbol{u}\,|_2 + (\vartheta - 1)\,|\,\nabla\cdot\boldsymbol{u}\,|_2\,.$$

Hypothesis $(5)_1$ implies then

$$|\,\rho'\,|_2 \leq \frac{8\,\nu k}{7\,m_\theta}\,|\,\nabla\theta'\,|_2 + \frac{8\,\mathrm{M}^2}{7\,\mathrm{R}m_\theta}(\vartheta - 1)\,|\,\nabla\cdot u\,|_2 + \frac{8\,\mathrm{M}^2 c}{7\,\mathrm{R}m_\theta}(1+2\,\mathrm{R}k^2\nu)\,|\nabla\boldsymbol{u}\,|_2\,. \tag{10}$$

We study, now, equations (7). To this end, employing $(6)_3$, Schwarz and Poincarè inequalities, we notice that the following relations hold true

$$\left\{\begin{aligned}
&\int_\Omega \theta\rho'\,\nabla\cdot\boldsymbol{u}\,\mathrm{d}x = -\int_\Omega \theta\,\{p'u\cdot\nabla\log\rho + \tfrac{1}{2}\,[(\nabla\cdot\hat{\boldsymbol{v}}/\rho) + (\hat{\boldsymbol{v}}\cdot\nabla\rho/\rho^2)]\,(\rho')^2\}\,\mathrm{d}x \leq \\
&\qquad\qquad \leq kk_1\,|\,\rho'\,|_2\,(\nu\,|\,\nabla\boldsymbol{u}\,|_2 + |\,\rho'\,|_2) \\
&\mathrm{F}_1 \leq \mathrm{R}\,(\nu b + k^2)\,|\,\rho'\,|_2\,|\,\nabla\boldsymbol{u}\,|_2 + \mathrm{R}k^2\nu\,|\,\nabla\boldsymbol{u}\,|_2^2 + \frac{\mathrm{R}k}{\mathrm{M}^2/\nu}\,|\,\nabla\theta'\,|_2\,|\,\nabla\cdot\boldsymbol{u}\,|_2 \\
&\mathrm{F}_2 = \mathrm{Pr}\left(\frac{\mathrm{R}k^2}{c_p/c_v} + 4\,k^2\right)\nu\,|\,\nabla\boldsymbol{u}\,|_2\,|\,\nabla\theta'\,|_2 + \mathrm{Pr}\,\nu\left(\frac{\mathrm{R}k^2}{c_p/\mathrm{R}^*} + 2k\,(\vartheta - 1)\right)\times \\
&\quad \times|\,\nabla\cdot\boldsymbol{u}\,|_2\,|\nabla\theta'|_2 + \mathrm{RPr}\,k^2\left(\frac{c_v + \mathrm{R}^*\nu}{c_p}\right)|\,\rho'|_2\,|\,\nabla\theta'|_2 + \mathrm{RPr}\,k^2\nu^2\mathrm{R}^*\,|\nabla\theta'|_2^2/c_p + \\
&\quad + \mathrm{Pr}\,2k\,|\,\nabla\boldsymbol{u}\,|_2^2 + \mathrm{Pr}\,k\,(\vartheta - 1)\,|\,\nabla\cdot\boldsymbol{u}\,|_2^2\,.
\end{aligned}\right. \tag{11}$$

Substituting estimate (11) in (7), we deduce

$$\left\{\begin{aligned}
&|\,\nabla\boldsymbol{u}\,|_2^2 \leq (\mathrm{R}kk_1/\mathrm{M}^2)\,|\,\rho'\,|_2^2 + \mathrm{R}\left[\frac{kk_1\,\nu}{\mathrm{M}^2} + \nu b + k^2\right]|\,\rho'\,|_2\,|\nabla\boldsymbol{u}\,|_2 + \\
&\quad + \mathrm{R}\nu\,k^2\,|\,\nabla\boldsymbol{u}\,|_2^2 + \frac{\sqrt{3}\,\mathrm{R}k\nu}{\mathrm{M}^2}\,|\,\nabla\theta'\,|_2\,|\,\nabla\boldsymbol{u}\,|_2 \\
&|\,\nabla\theta'\,|_2^2 \leq \mathrm{Pr}\left[\frac{\mathrm{R}k\,(c_v + \sqrt{3}\,\mathrm{R}^*)}{c_p} + 4 + 2\sqrt{3}\,(\vartheta - 1)\right]\nu k\,|\,\nabla\theta'\,|_2\,|\nabla\boldsymbol{u}\,|_2 + \\
&\quad + \mathrm{RPr}\,k^2\,\frac{c_v + \mathrm{R}^*\,\nu}{c_p}\,|\,\rho'\,|_2\,|\,\nabla\theta'\,|_2 + (\mathrm{RPr}\,k^2\,\nu^2\,\mathrm{R}^*/c_p)\,|\,\nabla\theta'\,|_2^2 + \\
&\quad + \mathrm{Pr}\,k\,(3\,\vartheta - 1)\,|\,\nabla\boldsymbol{u}\,|_2^2
\end{aligned}\right. \tag{12}$$

where we used the inequality [2] $|\,\nabla\cdot\boldsymbol{u}\,|^2 \leq 3\,|\nabla\boldsymbol{u}\,|^2$.

(2) Obviously, more accurate estimates can be obtained if we retain, on the left hand side of $(12)_1$, the term $(\vartheta - 1)\,|\,\nabla\cdot\boldsymbol{u}\,|_2$.

In these relations we employ (12) to obtain

$$
(13)\quad \begin{cases}
(1-\mathrm{R}\nu k^2)\,|\nabla \boldsymbol{u}|_2^2 \le a_4\,|\nabla\theta'|_2^2 + (k_1/m_\theta)\,a_2\,|\nabla \boldsymbol{u}|_2^2 + \\
\quad + \mathrm{R}\left\{\dfrac{8\nu k}{7m_\theta}\,[kk_1\nu\mathrm{M}^{-2} + \nu b + k^2] + \sqrt{3}\,k\nu\mathrm{M}^{-2}\right\}|\nabla\theta|_2\,|\nabla \boldsymbol{u}|_2 + \\
\quad + \dfrac{8m_\theta^{-1}\mathrm{M}^2}{7}\,(kk_1\nu\mathrm{M}^{-2} + \nu b + k^2)\,[\sqrt{3}\,(\vartheta-1) + c(1+2\mathrm{R}k^2\nu)]\,|\nabla \boldsymbol{u}|_2^2 \\
[1-(\mathrm{R}Pr\,k^2\nu^2\mathrm{R}^*/c_p)]\,|\nabla\theta'|_2^2 \le Pr\,b_3\,|\nabla \boldsymbol{u}|_2\,|\nabla\theta'|_2 + Pr\,b_2\,|\nabla \boldsymbol{u}|_2^2 + \\
\quad + \left[\dfrac{8}{7}\,\mathrm{R}Pr\,k^3\nu\,(c_v + \mathrm{R}^*\nu)/c_p\,m_\theta\right]|\nabla\theta'|_2^2 .
\end{cases}
$$

Since conditions (5) are assumed to hold, from (13) we deduce

$$
(14)\quad \begin{cases}
\left(\dfrac{5}{6} - \dfrac{\mathrm{M}^2}{m_\theta}\,a_1 - \dfrac{k_1}{m_\theta}\,a_2\right)|\nabla \boldsymbol{u}|_2^2 \le \mathrm{R}a_3\,|\nabla\theta'|_2\,|\nabla \boldsymbol{u}|_2 + \mathrm{R}a_4\,|\nabla\theta'|_2^2 \\
(1-\mathrm{R}Pr\,b_1)\,|\nabla\theta'|_2^2 \le Pr\,b_2\,|\nabla \boldsymbol{u}|_2^2 + Pr\,b_3\,|\nabla \boldsymbol{u}|_2\,|\nabla\theta'|_2 .
\end{cases}
$$

Employing again (5) into (14) and adding $(14)_1$ to $(14)_2$ times $\eta$ $(>0)$ we have

$$
(15)\quad \left(\frac{1}{2} - \eta\,Pr\,b_2\right)|\nabla \boldsymbol{u}|_2^2 + \left(\frac{\eta}{2} - \mathrm{R}a_4\right)|\nabla\theta'|^2 \le (\mathrm{R}a_3 + \eta\,Pr\,b_3)\,|\nabla \boldsymbol{u}|_2\,|\nabla\theta'|_2 .
$$

Applying, now, the Cauchy inequality in (15) we finally deduce

$$
(16)\quad \begin{aligned}
&\left[\frac{1}{2} - \eta\,\frac{Pr}{2}\,(2b_2 + b_3) - \frac{\mathrm{R}a_3}{2}\right]|\nabla \boldsymbol{u}|_2^2 + \\
&+ \left[\frac{\eta}{2} - \frac{\mathrm{R}}{2}\,(2a_4 + a_3) - \eta\,Pr\,\frac{b_3}{2}\right]|\nabla\theta'|_2^2 \le 0 .
\end{aligned}
$$

Choosing $\eta = 8\,\mathrm{R}\,(2a_4 + a_3)$ and employing (5) in (16), we obtain $\nabla \boldsymbol{u} = 0$ $\nabla\theta' = 0$ and consequently $\rho' = 0$. C.V.D.

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