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Some results on homotopy theory of modules

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Geometria. — *Some results on homotopy theory of modules.* Nota di HE ZHENG–XU, presentata (*) dal Socio E. MARTINELLI.

RIASSUNTO. — Seguendo le idee presentate nei lavori [1] e [2] si studiano le proprietà dei gruppi di i–omotopia per moduli ed omomorfismi di moduli.

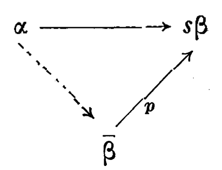
B. Eckmann and P. Hilton introduced and studied various homotopy groups of modules and of pairs (1) in [1] and [2]. They showed that many aspects of the homotopy theory of modules are similar to those based on topological spaces. In this note, some new results are presented. In section 1, the Ext groups of pairs are defined and general properties of pairs are studied. In section 2, we will show an exact sequence. Finally, in section 3, we will deal with fibre maps and give some "natural" homomorphisms of homotopy groups.

We will deal only with the i-homotopy; the p-homotopy can be presented dually. We will use implicitly the definitions given in [3, Ch. 13].

1. Let $i = (i^1, i^2) \in \mathrm{Hom}(\alpha, \alpha')$, where $\alpha : A_1 \to A_2$ and $\alpha' : A_1' \to A_2'$, define the *quotient pair* along i to be the pair $\bar{i} = \alpha'/\alpha : A_1'/\mathrm{Im}\, i^1 \to A_2'/\mathrm{Im}\, i^2$, where α'/α is induced by α'. If i is an inclusion of α in $\bar{\alpha}$, then we define the *suspension* of the pair α, denoted by $s\alpha$, to be $\bar{\alpha}/\alpha$ (along i) (2). Note that a suspension of the *pair* α is also a suspension of the *map* $\phi = \alpha$ defined in [3, p. 134], but the converse is not true. For any map $\phi : \alpha \to \beta$, we have an extension map $\bar{\phi} : \bar{\alpha} \to \bar{\beta}$ which, in turn induces a suspension map $S\phi : s\alpha \to s\beta$. In [3, Ch. 13], $\bar{\pi}(\alpha, \beta)$ is defined; we denote $\bar{\pi}_n(\alpha, \beta)$ for $\bar{\pi}(s^n \alpha, \beta)$. Any map $\phi : \alpha \to \alpha'$ induces homomorphisms of groups

$$\phi^* : \bar{\pi}_n(\alpha', \beta) \to \bar{\pi}_n(\alpha, \beta) \quad \text{and} \quad \phi_* : \bar{\pi}_n(\beta, \alpha) \to \bar{\pi}_n(\beta, \alpha').$$

Let us denote by $p : \bar{\beta} \to \bar{\beta}/\beta = s\beta$ the projection map, and we say that a map $\alpha \to s\beta$ is strongly i-null homotopic, if it can factor through $\bar{\beta}$:



Two maps $\alpha \to s\beta$ are said to be strongly i-homotopic if their difference is strongly i-null homotopic. We define *Ext group* of the pairs α, β, denoted by

(*) Nella seduta del 23 giugno 1983.

(1) A homomorphism of modules is said to be a pair, if it is considered as an object in the category of homomorphisms of modules. They are denoted by $\alpha, \beta, \cdots$.

(2) We denote by $\bar{\alpha}, \bar{\beta}, \cdots$ for some injective *pairs* containing $\alpha, \beta, \cdots$ respectively.

$\mathrm{Ext}(\alpha, \beta)$, to be the group of strongly i-homotopy classes of maps $\alpha \to s\beta$. Also define $\mathrm{Ext}^{n+1}(\alpha, \beta)$ to be $\mathrm{Ext}(\alpha, s^n\beta)$. Clearly, a map $\phi : \beta \to \beta'$ induces a homomorphisms $\phi^{\#} : \mathrm{Ext}^n(\alpha, \beta) \to \mathrm{Ext}^n(\alpha, \beta')$. Let $\omega = 0 : 0 \to A$ and let $\iota_n : S^{n-1}B \to \overline{S^{n-1}B}$ be the inclusion. We put $\mathrm{Ext}^{n+1}(\alpha, B) = \mathrm{Ext}(\alpha, \iota_n)$ and $\mathrm{Ext}^{n+1}(A, \beta) = \mathrm{Ext}(\omega, s^n\beta)$. It is easy to prove that $\mathrm{Ext}^{n+1}(\omega, B) = \mathrm{Ext}(A, \iota_n) = \mathrm{Ext}(\omega, \iota_n) = \mathrm{Ext}^{n+1}(A, B)$.

We say that a map $\phi : \alpha \to \beta$ is an i-homotopy equivalence if there is a map $\psi : \beta \to \alpha$ such that $\psi\phi \simeq_i 1_\alpha$ and $\phi\psi \simeq_i 1_\beta$, in this case we write $\alpha \simeq_i \beta$.

Note that the i-homotopy type of $s\alpha$ depends only on that of α.

THEOREM 1.1. *The following four statements about $\phi : \beta \to \beta'$ are equivalent:*

i) $\phi : \beta \simeq_i \beta'$;

ii) $\phi_* : \overline{\pi}(\alpha, \beta) \cong \overline{\pi}(\alpha, \beta')$, for any α ;

iii) $\phi^* : \overline{\pi}(\beta', \alpha) \cong \overline{\pi}(\beta, \alpha)$, for any α ;

iv) $\phi^{\#} : \mathrm{Ext}(\alpha, \beta) \cong \mathrm{Ext}(\alpha, \beta')$, for any α.

As a corollary, a pair β is injective if and only if $\beta \simeq_i 0$.

If $\alpha : A_1 \to A_2$, $\alpha' : A_1' \to A_2'$ are two pairs, their sum $\alpha \oplus \alpha'$ is then a pair $A_1 \oplus A_1' \to A_2 \oplus A_2'$.

PROPOSITION 1.2. *For any $\alpha, \alpha', \beta, \beta'$:*

i) $\overline{\pi}_n(\alpha \oplus \alpha', \beta) \cong \overline{\pi}_n(\alpha, \beta) \oplus \overline{\pi}_n(\alpha', \beta)$;

ii) $\overline{\pi}_n(\alpha, \beta \oplus \beta') \cong \overline{\pi}_n(\alpha, \beta) \oplus \overline{\pi}_n(\alpha, \beta')$;

iii) $\mathrm{Ext}^n(\alpha \oplus \alpha', \beta) \cong \mathrm{Ext}^n(\alpha, \beta) \oplus \mathrm{Ext}^n(\alpha', \beta)$;

iv) $\mathrm{Ext}^n(\alpha, \beta \oplus \beta') \cong \mathrm{Ext}^n(\alpha, \beta) \oplus \mathrm{Ext}^n(\alpha, \beta')$.

THEOREM 1.3. *$\phi : \alpha \to \beta$ is an i-homotopy equivalence if and only if there exist two injective pairs u and u' so that ϕ can be factored into:*

$$(*) \qquad \alpha \xrightarrow{\ i\ } u \oplus \alpha \xrightarrow{\ \phi'\ } u' \oplus \beta \xrightarrow{\ p\ } \beta$$

where i and p are the obvious maps, and ϕ' is an isomorphism of pairs.

Proof. We need prove only the necessity. Let ϕ be an i-homotopy equivalence. Let $i_1 = (i_1^1, i_1^2) : \alpha \to \overline{\alpha}$ be the inclusion, let $\lambda : \alpha \to \overline{\alpha} \oplus \beta$ be defined by $\lambda^1(a_1) = (i_1^1(a_1), \phi^1(a_1))$, $\lambda^2(a_2) = (i_1^2(a_2), \phi^2(a_2))$, for $a_1 \in A_1$, $a_2 \in A_2$.

Then λ is also an i-homotopy equivalence. Let $\mu : \overline{\alpha} \oplus \beta \to \alpha$ be its i-homotopy inverse, i.e., $\mu\lambda - 1_\alpha \simeq_i 0 : \alpha \to \alpha$, and λ is clearly an inclusion, so there is a map $\theta : \overline{\alpha} \oplus \beta \to \alpha$ such that $\theta\lambda = \mu\lambda - 1_\alpha$.

Let $u = (\overline{\alpha} \oplus \beta)/\alpha : U_1 = (\overline{A}_1 \oplus B_1)/\mathrm{Im}\,\lambda^1 \to U_2 = (\overline{A}_2 \oplus B_2)/\mathrm{Im}\,\lambda^2$ be the quotient pair. We have the exact sequences

$$0 \longrightarrow A_1 \xrightarrow{\lambda^1} \overline{A}_1 \oplus B_1 \xrightarrow{\tau^1} U_1 \longrightarrow 0$$

and

$$0 \longrightarrow A_2 \xrightarrow{\lambda^2} \overline{A}_2 \oplus B_2 \xrightarrow{\tau^2} U_2 \longrightarrow 0$$

where τ^1 and τ^2 are the projection maps.

By the proof of [3, Th. 13.17], $\overline{A}_1 \oplus B_1 \cong U_1 \oplus A_1$, $\overline{A}_2 \oplus B_2 \cong U_2 \oplus A_2$ and, if $\nu^1 : U_1 \to \overline{A}_1 \oplus B$, $\nu^2 : U_2 \to \overline{A}_2 \oplus B$ are inclusion maps, then $1_{\overline{A}_1 \oplus B_1} - \lambda^1 \theta^1 = \nu^1 \tau^1$ and $1_{\overline{A}_2 \oplus B_2} - \lambda^2 \theta^2 = \nu^2 \tau^2$. But, $(\overline{\alpha} \oplus \beta)\,\lambda^1 = \lambda^2 \alpha$ and $\alpha\theta^1 = \theta^2 (\overline{\alpha} \oplus \beta)$, which implies $(\overline{\alpha} \oplus \beta)\,\nu^1 \tau^1 = \nu^2 \tau^2 (\overline{\alpha} \oplus \beta) = \nu^2 u\, \tau^1$, hence $(\overline{\alpha} \oplus \beta)\,\nu^1 = \nu^2 u$, i.e. $\nu = (\nu^1, \nu^2)$ is a map of pairs $u \to \overline{\alpha} \oplus \beta$.

Now, let $\phi' : u \oplus \alpha \to \overline{\alpha} \oplus \beta$ be defined by ν and λ. Then ϕ' is an isomorphism and ϕ is represented by $(*)$, with $u' = \overline{\alpha}$. It remains to show that u is an injective pair. For this, we observe that $i = (\phi')^{-1} \lambda$ is an i-homotopy equivalence. By Proposition 1.2 and Theorem 1.1, we deduce $\overline{\pi}(\gamma, u) = 0$, for any pair γ, therefore u is injective.

2. For a pair α and a module B, we put $\overline{\pi}_n(\alpha, B) = \overline{\pi}_n(\alpha, \omega)$, where $\omega : 0 \to B$.

We have $\overline{\pi}_n(A, \omega) = \overline{\pi}_n(\iota_n, B) = \overline{\pi}_n(\iota_n, \omega) = \overline{\pi}_n(A, B)$, where $\iota_n : S^{n-1} A \to \overline{S^{n-1} A}$.

Let $\alpha : A_1 \to A_2$ be any pair, we denote A_0 for Ker α and A_3 for Coker α. If $\phi = (\phi^1, \phi^2) \in \mathrm{Hom}(\alpha, \beta)$, then ϕ^1 induces a map $\phi^0 : A_0 \to B_0$ and ϕ^2 induces a map $\phi^3 : A_3 \to B_3$.

Note that $\mathrm{Ker}(s\alpha) = S(\mathrm{Ker}\,\alpha)$ and $\mathrm{Coker}(s\alpha) = S(\mathrm{Coker}\,\alpha)$ and $(S\phi)^0 = S\phi^0 : SA_0 \to SB_0$, $(S\phi)^3 = S\phi^3 : SA_3 \to SB_3$.

THEOREM 2.1. *If* $A_0 = \mathrm{Ker}\,\alpha$ *is injective, then we have an exact sequence*:

$$\cdots \longrightarrow \overline{\pi}_n(\alpha, B) \xrightarrow{j^*} \overline{\pi}_n(A_2, B) \xrightarrow{\alpha^*} \overline{\pi}_n(A_1, B) \xrightarrow{\partial^*} \overline{\pi}_{n-1}(\alpha, B)$$
$$\cdots \to \overline{\pi}_1(A_2, B) \xrightarrow{\alpha^*} \overline{\pi}_1(A_1, B).$$

Proof. Firstly, we must define j^* and ∂^*.

Let $[x] \in \overline{\pi}_n(\alpha, \beta) = \overline{\pi}(s^n \alpha, \omega)$, $x = (x^1, x^2) : s^n \alpha \to \omega$, where $x^1 = 0 : S^n A_1 \to 0$ and $x^2 : S^n A_2 \to B$. Define $j^*[x] = [x^2]$. As for the definition of ∂^*, let us construct $\overline{s^{n-1}\alpha}$ and $s^n \alpha$ for $n \geq 1$, assuming that $\mathrm{Ker}(s^{n-1}\alpha)$ is injective. Let $i_n^1 : S^{n-1} A_1 \to \overline{S^{n-1} A_1}$, let $H = \{(i_n^1(a_1), -(s^{n-1}\alpha)(a_1))\,;\, a_1 \in S^{n-1} A_1\} \subseteq \overline{S^{n-1} A_1} \oplus S^{n-1} A_2$, and let $X = (\overline{S^{n-1} A_1} \oplus S^{n-1} A_2)/H$. Take i_n^2 to be the composition

$$S^{n-1}A_2 \hookrightarrow \overline{S^{n-1}A_1} \oplus S^{n-1}A_2 \longrightarrow (\overline{S^{n-1}A_1} \oplus S^{n-1}A_2)/H = X \hookrightarrow \overline{X}$$

and take $\overline{s^{n-1}\alpha} : \overline{S^{n-1}A_1} \to \overline{X}$ to be the composition

$$\overline{S^{n-1}A_1} \hookrightarrow \overline{S^{n-1}A_1} \oplus S^{n-1}A_2 \longrightarrow (\overline{S^{n-1}A_1} \oplus S^{n-1}A_2)/H = X \hookrightarrow \overline{X}\,.$$

Then i_n^2 is obviously an inclusion of $S^{n-1}A_2$ in $\overline{X}$ and $(\overline{s^{n-1}\alpha})\, i_n^1 = i_n^2 (s^{n-1}\alpha)$, so we can take $\overline{S^{n-1}A_2} = \overline{X}$. In this way, since Ker $(s^{n-1}\alpha)$ is injective, $\overline{s^{n-1}\alpha}$ is an injective pair, $i_n = (i_n^1, i_n^2) : s^{n-1}\alpha \to \overline{s^{n-1}\alpha}$ is an inclusion, and Ker $s^n\alpha = 0$ is injective for $s^n\alpha = \overline{s^{n-1}\alpha}/s^{n-1}\alpha$. For any $n \geq 2$, $\overline{s^{n-1}\alpha}$ is obviously a monomorphism of modules.

Now we define ∂^*. Let $[z] \in \pi_n(A_1, B) = \pi(i_n^1, \omega)$ $(n \geq 2)$ be represented by $(z^1, z^2) : i_n^1 \to \omega$. In the diagram:

$$\begin{array}{ccccc}
S^{n-1}A_2 & \xleftarrow{\;s^{n-1}\alpha\;} & S^{n-1}A_1 & \xrightarrow{\;z^1=0\;} & 0 \\
{\scriptstyle i_n^2}\big\downarrow & & {\scriptstyle i_n^1}\big\downarrow & & \big\downarrow{\scriptstyle \omega} \\
\overline{S^{n-1}A_2} & \underset{\dashrightarrow\,\theta\,}{\xleftarrow{\;\overline{s^{n-1}\alpha}\;}} & \overline{S^{n-1}A_1} & \xrightarrow{\;z^2\;} & \beta
\end{array}$$

$\overline{s^{n-1}\alpha}$ is monomorphism of modules, $\overline{S^{n-1}A_1}$ is injective, so there is a map $\theta : \overline{S^{n-1}A_2} \to \overline{S^{n-1}A_1}$ such that $\theta(\overline{s^{n-1}\alpha}) = 1_{\overline{S^{n-1}A_1}}$. Let $x^1 = z^1 = 0$, $x^2 = z^2\,\theta i_n^2$, then $x^2(s^{n-1}\alpha) = z^2\,\theta i_n^2 (s^{n-1}\alpha) = z^2\,\theta\,(\overline{s^{n-1}\alpha})\, i_n^1 = z^2\, i_n^1 = \omega z^1 = \omega x^1$, hence $x = (x^1, x^2) \in \mathrm{Hom}\,(s^{n-1}\alpha, \omega)$. Define $\partial^*([z]) = [x]$. The maps j^* and ∂^* are well-defined and they are homomorphisms of groups. We can verify directly that $\alpha^* j^* = 0$, $\partial^*\alpha^* = 0$ and $j^*\partial^* = 0$. It remains to show that Ker $\alpha_* \subseteq \mathrm{Im}\, j^*$, Ker $\delta^* \subseteq \mathrm{Im}\,\alpha^*$ and Ker $j^* \subseteq \mathrm{Im}\,\partial^*$. Let $[y] \in \mathrm{Ker}\,\alpha^*$ be represented by $y : S^n A_2 \to B$. Then, there is a map $\lambda : \overline{S^n A_1} \to B$ such that $\lambda i_{n+1}^1 = y(s^n\alpha)$. Then $(0, y - \lambda\theta\, i_{n+1}^2) \in \mathrm{Hom}\,(s^n\alpha, \omega)$, ($\theta : \overline{S^n A_2} \to \overline{S^n A_1}$ is such that $\theta(\overline{s^n\alpha}) = 1_{\overline{S^n A_1}}$) and $j^*([0, y - \lambda\theta i_{n+1}^2]) = [y]$, hence Ker $\alpha^* \subseteq \mathrm{Im}\, j^*$.

Let $[z] \in \mathrm{Ker}\,\partial^*$, where $z = (z^1 = 0, z^2) : i_n^1 \to \omega$. Then, there exists a map $(0, \mu) : \overline{s^{n-1}\alpha} \to \omega$, such that $\mu i_n^2 = z^2\,\theta i_n^2$. We get $(0, z^2\theta - \mu) \in$ $\in \mathrm{Hom}\,(i_n^2, \omega)$ and $\alpha^*([0, z^2\theta - \mu]) = [z]$, hence Ker $\partial^* \subseteq \mathrm{Im}\,\alpha^*$.

Lastly, let $[x] \in \mathrm{Ker}\, j^*$, with $x = (x^1 = 0, x^2) : s^n\alpha \to \omega$. Then, there exists a map $\eta : \overline{S^n A_2} \to B$ such that $\eta\, i_{n+1}^2 = x^2$. Then $(0, \eta(\overline{s^n\alpha})) \in$ $\in \mathrm{Hom}\,(i_{n+1}^1, \omega)$ and $\partial^*([0, \eta(\overline{s^n\alpha})]) = [x]$, so Ker $j^* \subseteq \mathrm{Im}\,\partial^*$. The proof is thus complete.

Proposition 2.2. *For any* $\alpha : A_1 \to A_2$, *let* $A_3 =$ *Coker* $\alpha = A_2/\alpha A_1$. *There is an isomorphism* $f : \pi_n(\alpha, B) \cong \pi_n(A_3, B)$. *If* $\phi : \alpha \to \alpha'$ *is a map, then* $f\phi^* = (\phi^3)^* f$, *where* $\phi^3 : A_3 \to A_3'$.

Proof. Let $x = (x^1, x^2) : s^n \alpha \to \omega$, $x^1 = 0$, $x^2(s^n \alpha) = \omega x^1 = 0$, so x^2 induces a map $x^3 : (S^n A_2)/\mathrm{Im}\, s^n \alpha \to B$. But $(S^n A_2)/\mathrm{Im}\, s^n \alpha = S^n A_3$, so we put $f([x]) = [x^3]$.

A straightforward proof shows that f is an isomorphism.

It is easy to see that, if we identify $\overline{\pi}_n(\alpha, B)$ with $\overline{\pi}_n(A_3, B)$ via f, then the homomorphism j^* in Theorem 2.1 is just the homomorphism induced by the "projection" map $j : A_2 \to A_2/\mathrm{Im}\,\alpha = A_3$. So, if Ker α is injective, we have the exact sequence:

$$\cdots \to \overline{\pi}(A_3, B) \xrightarrow{j^*} \overline{\pi}_n(A_2, B) \xrightarrow{\alpha^*} \overline{\pi}_n(A_1, B) \xrightarrow{\partial^*} \overline{\pi}_{n-1}(A_3, B) \to \cdots$$

If Ker α, Ker α' are injective, observe that a map $\phi : \alpha \to \alpha'$ (as well as a map $\gamma : B \to B'$) induces a commutative diagram of such sequences. From this diagram, it follows that $S\phi^3$ is an i-homotopy equivalence if $S\phi^1$ and $S\phi^2$ are.

Similarly, $S^2\phi^1$ (or $S^2\phi^2$) is an i-homotopy equivalence if $S\phi^2$ (or $S^2\phi^1$) and $S\phi^3$ are.

3. Recall that a map $\beta : B_1 \to B_2$ is a fibre map if we may lift any map $\mu : I \to B_2$ to B_1, where I is any injective. There is an "excision" homomorphism $\varepsilon : \overline{\pi}_{n-1}(A, B_0) \to \overline{\pi}_n(A, \beta)$, which is an isomorphism if $\beta : B_1 \to B_2$ is a fibre map, where $B_0 = \mathrm{Ker}\,\beta$ is the fibre of β.

Let $p : \overline{B} \to SB = \overline{B}/B$ be the projection map.

PROPOSITION 3.1. *If p is a fibre map, then* $\overline{\pi}(A, B) \cong \overline{\pi}_1(A, SB)$.

Proof. If p is a fibre map, then $\varepsilon : \overline{\pi}(A, B) \cong \overline{\pi}_1(A, p)$. From [3, Theorem 13.15], we have the exact sequence for p:

$$\cdots \to \overline{\pi}_1(A, \overline{B}) \xrightarrow{p^*} \overline{\pi}_1(A, SB) \xrightarrow{J} \overline{\pi}_1(A, p) \xrightarrow{\partial} \overline{\pi}(A, \overline{B}) \to \cdots$$

But $\overline{\pi}_1(A, \overline{B}) = 0$ and $\overline{\pi}(A, \overline{B}) = 0$, so $\overline{\pi}_1(A, p) \cong \overline{\pi}_1(A, SB)$. We conclude then $\overline{\pi}(A, B) \cong \overline{\pi}_1(A, SB)$.

Note that the homomorphism that carries $[x] \in \overline{\pi}(A, B)$ in $[Sx] \in \overline{\pi}_1(A, SB) = \overline{\pi}(SA, SB)$ is an isomorphism that differs from the isomorphism of Proposition 3.1 only by the sign.

The following proposition doesn't hold in the topological case, so the homotopy theory of modules differs from the homotopy theory of topological space.

PROPOSITION 3.2. $B \simeq_i 0$ *if and only if* p *is a fibre map and* $SB \simeq_i 0$.

Proof. The necessity is obvious. The converse is a consequence of Proposition 3.1.

Note that if B is a module over a principal ring, then since $SB \simeq_i 0$, B is injective if and only if p is a fibre map.

For any pair α, denote by α_0 the inclusion $A_0 = \operatorname{Ker} \alpha \hookrightarrow A_1$.

PROPOSITION 3.3. *There is a homomorphism of groups*

$$g : \overline{\pi}(\alpha, \beta) \to \overline{\pi}(\alpha_0, \beta_0) \oplus \overline{\pi}(A_3, B_3).$$

g is a monomorphism if the projection map $q : B_2 \to B_3$ *is a fibre map and g is an isomorphism if both the projection maps* $p : A_2 \to A_3$ *and* $q : B_2 \to B_3$ *have right inverses.*

Proof. The construction of g can be seen from the following diagram:

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\overline{A}_0 & \longleftarrow & A_0 & \xrightarrow{x^0} & B_0 \\
\downarrow & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \beta_0} \\
\overline{A}_1 & \longleftarrow & A_1 & \xrightarrow{x^1} & B_1 \\
{\scriptstyle \overline{\alpha}}\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\
\overline{A}_2 & \longleftarrow & A_2 & \xrightarrow{x^2} & B_2 \\
\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\
\overline{A}_3 & \longleftarrow & A_3 & \xrightarrow{x^3} & B_3 \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

Explicitly, $g([x^1, x^2]) = ([x^0, x^1], [x^3])$. The rest is deduced from this diagram and the construction given in [3, Prop. 13.13].

There are also "natural" homomorphisms of groups:

i) $h_1 : \overline{\pi}(\alpha, \beta) \to \overline{\pi}(A_1, B_1) \oplus \overline{\pi}(A_2, B_2)$;

ii) $h_2 : \overline{\pi}(\alpha, \beta) \to \overline{\pi}(A_1, B_1) \oplus \overline{\pi}(A_3, B_3)$.

PROPOSITION 3.4. h_2 *is monomorphism if* $\operatorname{Ker} \alpha$ *is injective and* $q : B_2 \to B_3$ *is a fibre map*;

h_2 *is an isomorphism if* $A_2 \cong A_1 \oplus A_3$, α *is the inclusion* $A_1 \to A_1 \oplus A_3$ *and* $q : B_2 \to B_3$ *has a right inverse.*

Examples. i) If $\alpha : A_1 \to A_2 = A_1 \oplus A_3$ and $\beta : B_1 \to B_2 = B_1 \oplus B_3$ are inclusions, then $\overline{\pi}(\alpha, \beta) \cong \overline{\pi}(A_1, B_1) \oplus \overline{\pi}(A_3, B_3)$.

ii) If α is an inclusion of an injective module A_1 in A_2 and if $\beta = \omega : 0 \to B$, then we deduce $\overline{\pi}(\alpha, B) \cong \overline{\pi}(A_2/A_1, B)$, but clearly $A_2/A_1 \simeq_i A_2$, so we get $\overline{\pi}(\alpha, B) \cong \overline{\pi}(A_2, B)$.

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