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The spectrum of the Laplace operator for a spherical space form

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Geometria. — *The spectrum of the Laplace operator for a spherical space form.* Nota di GR. TSAGAS, presentata (*) dal Socio G. ZAPPA.

RIASSUNTO. — Si determina lo spettro di un operatore di Laplace di una « spherical space form » (M, g) e si studia l'influenza di tale spettro su (M, g).

1. Introduction. Let (M, g) be a compact Riemannian manifold whose sectional curvature is a positive constant. A Riemannian manifold with this property is called a spherical space form.

Let $C^\infty(M)$ be the algebra of all differential functions on M. The Laplace operator Δ is a linear operator on the algebra $C^\infty(M)$. The set of all eigenvalues of Δ is called spectrum of Δ on (M, g) and denoted by $Sp(M, g)$.

The aim of the present paper is to determine the $Sp(M, g)$, when (M, g) is a spherical space form. Subsequently we study the influence of $Sp(M, g)$ on the geometry of (M, g).

The whole paper contains four paragraphs. Each of them is analysed as follows.

The second paragraph deals with spherical space forms and some properties. These properties are useful for the next paragraph.

The spectrum of the Laplace operator for lens spaces whose fundamental group has order 5 are computed in the third paragraph.

The last paragraph deals with the coefficients of the Minakshisundaram-Pleijel-Gaffney asymptotic expansion for the lens spaces which have been considered in the third paragraph.

2. Let (M, g) be a compact connected Riemannian manifold of dimension n. The following theorem ([10]) is valid:

THEOREM 2.1. *Let* (M, g) *be a spherical space form of dimension* n. *Then* $M = S^n/\Gamma$, *where* Γ *is a fixed point free finite subgroup of* $0(n+1)$. *We distinguish three cases, (i) if* $\Gamma = \{1_{n+1}\}$, *then* $M = S^n$, *(ii) if* $\Gamma = \{\pm 1_{n+1}\}$, *the* $M = \mathbf{P}^n(\mathbf{R})$ *the real projective space, (iii) if the order of* Γ *is great than two, then* $n = 2m - 1$.

Now we put the following problem.

(*) Nella seduta del 23 giugno 1983.

PROBLEM 2.2. *Let* $(S^n/\Gamma, g_0/\Gamma)$ *be a spherical space form. Compute the spectrum of this Riemannian manifold, that is* $Sp(S^n/\Gamma, g_0/\Gamma) = Sp(S^n/\Gamma)$.

If $\Gamma = \{1_{n+1}\}$, then $(S^n/\Gamma, g_0/\Gamma)$ is the standard sphere (S^n, g_0) for which the spectrum is given by ([1]).

$\Gamma = \{\pm 1_{n+1}\}$, then $(S^n/\Gamma, g_0/\Gamma)$ is the real projective space $(\mathbf{P}^n(\mathbf{R}), g_0')$, whose spectrum is given by ([1]).

Now we must study the cases where the order of Γ is greater than two. Therefore the dimension n is odd, that is $n = 2m - 1$.

From now on we consider odd dimensional spherical space form, i.e.

$$(S^{2m-1}/\Gamma, g_0/\Gamma).$$

For this spherical space form we have the natural projection π of S^{m2-1} into S^{2m-1}/Γ, which induces the injective map π^*.

$$\pi^* : C^\infty(S^{2m-1}/\Gamma) \to C^\infty(S^{2m-1}).$$

In order to compute the $Sp(S^{2m-1}/\Gamma, g_0/\Gamma)$ we must associate to the spherical space form S^{2m-1}/Γ a special function which is called generating function which is defined as follows.

DEFINITION 2.3. The generating function denoted by $F_\Gamma(z)$ and associated with the spectrum of $\overline{\Delta}$ on $M = S^{2m-1}/\Gamma$ is defined as follows

$$F_\Gamma(z) = \sum_{k=0}^{\infty} (\dim H_\Gamma^k)\, z^k \tag{2.1}$$

where H_Γ^k are the eigenfunctions on S^{2m-1} with eigenvalues of order k which are invariant by the action of the group Γ.

The function $F_\Gamma(z)$ defined by (2.1) is a complex function of one variable. This plays an important role in the theory of the spectrum of the Laplace operator on spherical space forms.

The following theorem has been proved:

THEOREM 2.4. *Let* $M = S^{2m-1}/\Gamma$ *be a spherical space form. The generating function* $F_\Gamma(z)$ *of* $M = S^{2m-1}/\Gamma$ *on the unit disc* $D^1 = \{z \in \mathbf{C} / |z| < 1\}$ *converges to the function*

$$F_\Gamma(z) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{1 - z^2}{\operatorname{Det}(1 - \gamma^2 z)} \tag{2.2}$$

where $|\Gamma|$ *denotes the order of* Γ.

The generating function can also be given by the following theorem.

THEOREM 2.5. *Consider a spherical space form* $M = S^{2n-1}/\Gamma$. *The generating function can be written as follows*

$$(2.3)\qquad F_\Gamma(z) = \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} \frac{1-z^2}{\sum_{t=1}^{l} (z-\delta^{q}t)(z-\delta^{-q})}.$$

Relation (2.3) implies that the generating function $F_\Gamma(z)$ can be considered as a meromorphic function on the whole complex plane **C** and its poles are on the unit circle $S^1 = \{z \in \mathbf{C} / |z| = 1\}$.

This meromorphic extension of the generating function to the complex plane C is unique.

Let Γ be a cyclic subgroup of $SO(2m)$ of order q. We denote by γ a generator of Γ. Then γ is conjugate in $SO(2m)$ to the element

$$(2.4)\quad \gamma' = \begin{bmatrix} R\left(\frac{q_1}{q}\right) & & \\ & \ddots & \\ & & R\left(\frac{q_m}{q}\right) \end{bmatrix}, \text{ where } R(\theta) = \begin{bmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{bmatrix}$$

and $q_1, \cdots, q_m$ are integers prime to q.

The eigenvalues of γ are the same as the eigenvalues of γ'. If we set:

$$(2.5)\qquad \delta = \exp \frac{2\pi\sqrt{-1}}{q} = e^{\frac{2\pi i}{q}}$$

the eigenvalues of γ can be written as, $\delta^{q_1}, \delta^{-q_1}, \cdots, \delta^{q_l}, \delta^{-q_l}$.

DEFINITION 2.6. Let S^{2m-1}/Γ be a spherical space form, where Γ is a cyclic group of order q. Let γ be the generator of Γ whose eigenvalues are $\exp\sqrt{-1}\,(q_i/q)$, $\exp 2\pi\sqrt{1-}\,(-q_i/q)$, $i = 1, 2, \cdots, m$. Then M is called lens space and denoted by $L_{2m-1}(q : q_1, \cdots, m) = L(q : q_1, \cdots, q_m)$.

The following theorem has been proved.

THEOREM 2.7. *Let* $L(q : q_1, \cdots, q_n) = S^{2m-1}/\Gamma$ *be a lens space. The generating function* $F_\Gamma(z)$ *of* $L(q : q_1, \cdots, q_m)$ *is given by*

$$(2.6)\qquad F_\Gamma(z) = \frac{1}{q} \sum_{l=0}^{q-1} \frac{1-z^2}{\prod_{i=1}^{m} (1-\delta^{lq_i} z)(1-\delta^{-lq_i} z)}.$$

3. We assume that the group Γ which determines the spherical space form $M = S^{2m-1}/\Gamma$ is a cyclic group of order 5.

Let γ be a generator of the cyclic group Γ.

It is known that γ is conjugate in SO $(2\,m)$ to the element

$$\gamma' = \begin{bmatrix} \mathrm{R}\left(\frac{q_1}{5}\right) & & \\ & \ddots & \\ & & \mathrm{R}\left(\frac{q_m}{5}\right) \end{bmatrix} \tag{3.1}$$

where $q_1, \cdots, q_m$ are prime to 5 and take values 1 or 2 or 3 or 4.

The eigenvalues of γ are the same as the eigenvalues of γ'.

We set:

$$\delta = \exp \frac{2\,\pi\sqrt{-1}}{5} = e^{\frac{2\pi i}{5}} = \cos \frac{2\,\pi}{5} + i \sin \frac{2\,\pi}{5} \,. \tag{3.2}$$

From (3.2) we obtain

$$\delta^{-1} = \bar{\delta}\,,\ \delta^{-2} = \bar{\delta}^2\,,\ \delta^3 = \delta^{-2}\,,\ \delta^{-3} = \delta^2\,,\ \delta^4 = \delta^{-1}\,,\ \delta^{-4} = \delta\,. \tag{3.3}$$

From (3.2) and (3.3) we deduce that the only values of $q_1, \cdots, q_m$ which must be considered are 1 or 2.

THEOREM 3.1. *Let* $\mathrm{L}(5 : q_1, \cdots, q_m) = \mathrm{S}^{2m-1}/\Gamma$ *be a lens space, where* Γ *is a cyclic group of order* 5. *The generating function of the lens space is given by the formula*

$$\begin{aligned} \mathrm{F}_\Gamma(z) = \frac{(1-z^2)}{5} & \left[\left[\binom{2\,m-1}{0} + \binom{2\,m}{1} z + \cdots + \binom{2\,m+l-1}{l} z^l + \cdots\right] + \right. \\ & + 2\left[\binom{p-1}{0} + \binom{p}{1} z^5 + \cdots + \binom{p+l-1}{l} z^{5l} + \cdots\right] \cdot \\ & \cdot \left[2\,(1+z^2)^k - \binom{k}{1}(\alpha+\beta)(1+z^2)^{k-1} z + \right. \\ & \left. + \binom{k}{2}(\alpha^2+\beta^2)(1+z^2)^{k-2} z^2 - \cdots + (-1)^k (\alpha^k+\beta^k) z^k\right] \cdot \\ & \left.\cdot \left[\binom{p}{0} - \binom{p}{1} z + \cdots + (-1)^p \binom{p}{p} z^p\right]\right], \end{aligned}$$

where $\alpha = \delta + \delta^{-1}$ *and* $\beta = \delta^2 + \delta^{-2}$ *and* $\delta, \delta^{-1}, \delta^2, \delta^{-2}$ *are the roots of* $z^5 - 1 = 0$ *different from* 1.

Proof. For the lens space

$$\mathrm{L}(5 : q_1, \cdots, q_m) = \mathrm{S}^{2m-1}/\Gamma \tag{3.4}$$

where $q_1, \cdots, q_m$ are one or two, we assume that p numbers of them is 1 and $m-p$ are 2. Under this assumption we obtain that the generating function given by (2.6) takes the form

$$
\begin{aligned}
(3.5) \qquad F_\Gamma(z) = \frac{1-z^2}{5} \Bigg[& \frac{1}{(1-z)^{2m}} + \\
& + \frac{1}{(\delta - z)^p (\delta^{-1} - z)^p (\delta^2 - z)^{m-p} (\delta^{-2} - z)^{m-p}} + \\
& + \frac{1}{(\delta^2 - z)^p (\delta^{-2} - z)^p (\delta - z)^{m-p} (\delta^{-1} - z)^{m-p}} + \\
& + \frac{1}{(\delta - z)^p (\delta^{-1} - z)^p (\delta^2 - z)^{m-p} (\delta^{-2} - z)^{m-p}} + \\
& + \frac{1}{(\delta^2 - z)^p (\delta^{-2} - z)^p (\delta - z)^{m-p} (\delta - z)^{m-p}} \Bigg]
\end{aligned}
$$

which after some calculations and setting $k = 2p - m$ takes the form

$$
\begin{aligned}
(3.6) \quad F_\Gamma(z) = \frac{1-z^2}{5} \Bigg[& \frac{1}{(1-z)^{2m}} + 2 \frac{(1-z)^p}{(1-z^5)^p} \Big[2(1+z^2)^k - \\
& - \binom{k}{1} (\alpha + \beta)(1+z^2)^{k-1} z + \binom{k}{2} (\alpha^2 + \beta^2)(1+z^2)^{k-2} z^2 - \\
& - \binom{k}{3} (\alpha^3 + \beta^3)(1+z^2)^{k-3} z^3 + \cdots + \\
& + (-1)^{k-1} \binom{k}{k-1} (\alpha^{k-1} + \beta^{k-1})(1+z^2) z^{k-1} + (-1)^k (\alpha^k + \beta^k) z^k \Big] \Bigg]
\end{aligned}
$$

$$
(3.7) \quad \alpha + \beta = -\binom{1}{0}, \quad \alpha^2 + \beta^2 = 2\binom{2}{1} - \binom{2}{0}, \quad \alpha^3 + \beta^3 = -\binom{3}{1} - \binom{3}{0}
$$

$$
(3.8) \quad \alpha^4 + \beta^4 = 2\binom{4}{2} - \binom{4}{1} - \binom{4}{0}, \quad \alpha^5 + \beta^5 = 4 - \binom{5}{1} - \binom{5}{2}.
$$

From the above we conclude that

$$
(3.9) \quad \alpha^v + \beta^v = \begin{cases} 2\binom{v}{s} - \binom{v}{s-1} - \binom{v}{s-2} - \cdots - \binom{v}{1} - \binom{v}{0} & \text{if } v = 2s \\ 4 - \binom{v}{1} - \binom{v}{2} - \cdots - \binom{v}{\left[\frac{5t}{2}\right]} & \text{if } v = 5t \\ -\binom{v}{0} - \binom{v}{1} - \binom{v}{2} - \cdots \binom{v}{s} & \text{if } v = 2s+1. \end{cases}
$$

The following formulae can be easily proved

$$\frac{1}{(1-z)^{2m}} = \binom{2m-1}{0} + \binom{2m}{1} z + \binom{2m+1}{2} z^2 + \cdots + \binom{2m+l-1}{l} z^l + \cdots \tag{3.10}$$

$$\frac{1}{(1-z^5)^p} = \binom{p-1}{0} + \binom{p}{1} z^5 + \binom{p+1}{2} z^{10} + \cdots + \binom{p+l-1}{l} z^{5l} + \cdots . \tag{3.11}$$

Formula (3.6) by means of (3.7) and (3.8) takes the form

$$\begin{aligned} F_\Gamma(z) = \frac{(1-z^2)}{5} \Bigg[\Bigg[& \binom{2m-1}{0} + \binom{2m}{1} z + \cdots + \\ & + \binom{2m+l-1}{l} z^l + \cdots \Bigg] + 2 \Bigg[\binom{p-1}{0} + \binom{p}{1} z^5 + \cdots + \\ & + \binom{p+l-1}{l} z^{5l} + \cdots \Bigg] \cdot \Bigg[2(1+z^2)^k - \\ & - \binom{k}{1} (\alpha+\beta)(1+z^2)^{k-1} z + \binom{k}{2} (\alpha^2+\beta^2)(1+z^2)^{k-2} z^2 - \cdots + \\ & + (-1)^k (\alpha^k+\beta^k) z^k \Bigg] \Bigg[\binom{p}{0} - \binom{p}{1} z + \cdots + (-1)^p \binom{p}{p} z^p \Bigg] \Bigg] \end{aligned} \tag{3.12}$$

where $\alpha^v + \beta^v$, $v = 0, 1, \cdots, k$ are given by the formula (3.9) which depends on the form of v.

This completes the proof of the theorem.

From formula (3.12) we conclude that the generating function $F_\Gamma(z)$ of the lens space given by (3.4) takes the form

$$F_\Gamma(z) = 1 + p_1 z + p_2 z + \cdots + p_l z^l + \cdots, \quad P_i \in \mathbf{N} \qquad i = 1, 2, \cdots \tag{3.13}$$

and depends on the number k and can be completely determined.

Now we can state the following theorem

THEOREM 3.2. *Consider the lens space* $L_{2m-1}(5 : q_1, \cdots, q_m) = S^{2m-1}/\Gamma$, *where* Γ *is a cyclic group of order* 5. *The spectrum of this les space is given by*

$$\begin{aligned} Sp(L_{2m-1}(5 : q_1, \cdots, q_m)) = \\ = \{0 < \lambda_1 = \cdots = \lambda_1 < \lambda_2 = \cdots = \lambda_2 < \cdots < \infty\} \end{aligned} \tag{3.14}$$

where

$$(3.15) \qquad \lambda_i = 1\,(2m + i - 2)$$

and the multiplicity of λ_i *is* p_i *given by* (3.13).

Proof. From the form of the generating function of the lens space

$$(3.16) \qquad L_{2m-1}(5 : q_1, \cdots, q_m) = S^{2m-1}/\Gamma$$

which is given by (3.13) we conclude that this lens space has all the eigenvalues of the sphere, but the multiplicity of the eigenvalue λ_i is p_i given by (3.13) different from the multiplicity of λ_i when it is considered as eigenvalue for the sphere S^{2m-1}.

Let us study the special case $k=0$ and $m=2p$. In this case the generating function given by (3.12) of this lens space becomes

$$(3.17) \qquad F_\Gamma(z) = 1 + \delta_1 z + \delta_2 z^2 + \cdots + \delta_{5l+\sigma} z^{5l+\sigma} + \cdots$$

where

$$(3.18) \qquad \delta_1 = \frac{1}{5}\left\{\binom{2m}{1} - 4\binom{m-1}{0}\binom{v}{1}\right\}, \quad \delta_{5l} = \frac{1}{5}\{\gamma_{5l} - \gamma_{5(l-1)+3}\},$$

$$(3.19) \qquad \delta_{5l+1} = \frac{1}{5}\{\gamma_{5l+1} - \gamma_{5(l-1)+4}\}, \quad \delta_{5l+2} = \frac{1}{5}\{\gamma_{5l+2} - \gamma_{5l}\}$$

$$(3.20) \qquad \delta_{5l+3} = \frac{1}{5}\{\gamma_{5l+3} - \gamma_{5l+1}\}, \quad \delta_{5l+4} = \frac{1}{5}\{\gamma_{5l+4} - \gamma_{5l+2}\}$$

where the coefficients $\gamma_{5l+\sigma}$ are given by

$$(3.21) \qquad \gamma_{5l+\sigma} = \binom{2m+5l+\sigma-1}{5l+\sigma} + 4\sum_{t=0}^{l}(-1)^{5l+\sigma}\binom{m+p-1}{t}\binom{p}{5l-5t+\sigma}$$

$$l = 0, 1, 2, \cdots \qquad \sigma = 0, 1, 2, 3, 4, \cdots$$

and

$$(3.22) \qquad \binom{p}{5l-5t+\sigma} = 0, \quad \text{if } 5l-5t+\sigma > p, \quad \binom{p}{\sigma} = 0,$$

$$\text{if } \sigma > p, \quad \text{when } t=0 \text{ and } l=0.$$

THEOREM 3.3. *Let* $L_{2m-1}(5 : q_1, \cdots, \sigma_m)$ *be a lens space where half of the numbers of* $q_1, \cdots, q_m$ *are* 1 *and the others are* 2. *Then the generating function of this lens space is given by* (3.17).

From theorem 3.3. we obtain;

THEOREM 3.4. *Consider the lens space* $L_{4p-1}(5:1,\cdots,1,2,\cdots,2)$ (1 *appear p times*) *Then the spectrum of this is given by*

$$Sp(L_{4p-1}(5:1,\cdots,1,2,\cdots,2) =$$
$$=\{0<\lambda_1=\cdots=\lambda_1<\lambda_2=\cdots=\lambda_2<\cdots<\infty\}$$

where

$$\lambda_i = i(4p+i-2)$$

and the multiplicity of λ_i *is given by the formulas* (3.18), (3.19) *and* (3.20).

4. The partition function f_M of a Riemannian manifold M is defined by

$$f_M(t)=\sum_{i=0}^{\infty} m_i e^{-\lambda_i t} \quad t>0 \quad \lambda_i \in Sp(M,g),\ m_i = \text{mult}(\lambda_i),\ n=\dim M. \tag{4.1}$$

Then the Minakshisundaram–Pleijel Gaffney asymptotic expansion is

$$f_M(t) \underset{t\to 0}{\sim} +(4\pi t)^{-n/2}(\alpha_0+\alpha_1 t+\alpha_2 t^2+\cdots), \tag{4.2}$$
$$\text{where} \quad \alpha_q=\alpha_q(M)=\int_M u_q\, dM, \qquad q=0,1,\cdots.$$

For S^{2m-1} the coefficients $\alpha_q(S^{2m-1})$ are given by ([2])

$$\alpha_0(S^1)=2\pi,\ \alpha_q(S^1)=0,\ q\geq 1,\ \alpha_q(S^3)=\frac{16\pi^2}{q!\,4^q}, \tag{4.3}$$

$$\alpha_q(S^{2m-1})=\frac{2^{m-1}\pi^m}{(2m-2)!}\sum_{k=0}^{q}\frac{(m-1)^{q-\frac12}}{q!}\alpha_{m-1-k,m}\,b_{m-1-k}\,4^{m-\frac12-2k} \tag{4.4}$$
$$\text{if} \quad m\geq 3,\ q<m.$$

$$\alpha_q(S^{2m-1})=\frac{2^{m-1}\pi^m}{(2m-2)!}\sum_{k=q-m}^{q}\frac{(m-1)^{q-\frac12}}{k!}\alpha_{q-k-1,m}\,b_{m-k-1}\,4^{m-\frac12-2k} \tag{4.5}$$
$$\text{if} \quad m\geq 3,\ q\geq m,$$

where the numbers b_j and $\alpha_{k,n}$ are defined by

$$b_0-1,\ b_j=\frac12\,\frac32\cdots\frac{2j-1}{2},\ j=1,2,\cdots,\ \prod_{j=0}^{m-2}(t^2-j^2)=\sum_{k=0}^{m-1}\alpha_{k,m}t^{2k}. \tag{4.6}$$

Now, since S^{2m-1} and $L_{2m-1}(5:q_1,\cdots,q_m)$ are locally isometric the function u_q for S^{2m-1} and corresponding function $\bar u_q$ for $L_{2m-1}(5:q_1,\cdots,q_m)$ are related by $\bar u_q \circ \pi = u_q$, where π is the quotient map. We know that

$$\pi: S^{2m-1}\to L_{2m-1}(5:q_1,\cdots,q_m) \quad \text{is a 5-fold cover.} \tag{4.7}$$

Therefore we have

$$(4.8) \qquad \alpha_q\,(L_{2m-1}\,(5 : q_1 , \cdots , q_m)) = \frac{1}{5}\,\alpha_q\,(S^{2m-1})\,, \qquad q = 0\,,1\,,2\,,\cdots .$$

From the above we obtain the theorem

THEOREM 4.1. *The coefficients* $\alpha_q\,(L_{2m-1}\,(5 : q_1 , \cdots , q_m))$ *in the Minakshisundaram–Pleijel–Gaffney asymptotic expansion of the partition function*

$$f_{L_{m2-1}}\,(5 : q_1 , \cdots , q_m)$$

are defined by (4.8) *where each* $\alpha_q\,(S^{2m-1})$ *is given by* (4.3) *for* $m = 1\,,2$ *and by* (4.4) *and* (4.5) *for* $m \geq 2$.

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