Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Interpolation problems in cones. Nota II

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **74** (1983), n.6, p. 331–335.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1983_8_74_6_331_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1983.

Analisi matematica. — *Interpolation problems in cones.* Nota II di CARLOS A. BERENSTEIN e DANIELE STRUPPA, presentata (*) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Si estendono qui i risultati della nota precedente al caso di varietà non discrete. Ciò viene utilizzato per ottenere un teorema di rappresentazione per soluzioni di sistemi di equazioni di convoluzione in spazi di funzioni olomorfe in coni.

4. THE NON-DISCRETE CASE

In this section we wish to introduce the concept of a slowly decreasing vector valued function ρ when the associated variety V of common zeros of the components of ρ is not discrete. We are still considering everything in the space $\mathrm{Exp}_c(0, \Gamma)$. We follow [2], section 5 with the modifications that section 3 above suggests. Let then $\Gamma \subseteq \mathbf{C}^n$ be an open cone as before and let $\rho = (\rho_1, \cdots, \rho_m)$, $1 \leq m \leq n$ be an m-tuple of functions in $\mathrm{Exp}_c(0, \Gamma)$. (The case $m = n$ has been dealt with in section 3). Let $\mathscr{L}$ be a family of m-dimensional affine subspaces of $\mathbf{C}^n$ such that

$$\bigcup_{L \in \mathscr{L}} (L \cap \Gamma) \supseteq \{z \in \Gamma : \rho_1(z) = \cdots = \rho_m(z) = 0\} . \tag{4}$$

DEFINITION 4.1. We say that $\rho = (\rho_1, \cdots, \rho_m)$ is slowly decreasing if there exists a family $\mathscr{L}$, an exhaustion $\{\Gamma_k\}_{k \geq 1}$ of Γ, and two sequences of real numbers $\{\varepsilon_k\}$, $\{\alpha_k\}$, $\varepsilon_k > 0$, $\alpha_k > 0$ such that for every $L \in \mathscr{L}$ and every k, the relatively open set

$$S = S(\rho ; L, \varepsilon_k, \alpha_k) = \{z \in L \cap \Gamma(k) : |\rho(z)| < \varepsilon_k e^{-\alpha_k |z|}\}$$

has all its connected components of uniformly bounded diameters.

One can now introduce the concepts of good open sets, good refinements and almost parallel families as done in [2], section 5. The main difference lies in the fact that the space $\mathrm{Exp}_c(0, \Gamma)$ does not coincide with the space $A_p(\Gamma)$, $p(z) = |z|$, defined by

$$A_p(\Gamma) = \{f \in H(\Gamma) : \exists A, B > 0 \quad \text{such that} \quad |f(z)| \leq A e^{Bp(z)} \, \forall z \in \Gamma\} .$$

Hence, if C is a good family (of open sets) for the slowly decreasing map ρ, the double complex $\mathscr{A}_q^r = \mathscr{A}_q^r(C)$ is slightly different from the one introduced in [2].

(*) Nella seduta del 14 maggio 1983.

We say that $\gamma \in \mathscr{A}^r$, if it is an alternating function on $C \times \cdots \times C$ (($r+1$)-times) such that $\gamma(\Omega_0, \cdots, \Omega_r) \in \mathrm{H}(\Omega_0 \cap \cdots \cap \Omega_r)$ and satisfies the following condition: for every $\varepsilon > 0$, $k \geq 1$, there is a constant $\mathrm{A}_{\varepsilon,k} > 0$ for which

$$|\gamma(z)| \leq \mathrm{A}_{\varepsilon,k}\, e^{\varepsilon|z|} \qquad \text{on} \quad \Omega_0 \cap \Omega_1 \cap \cdots \cap \Omega_r \subset \Gamma(k).$$

One defines, for an index l $(1 \leq l \leq m)$

$$\mathscr{A}^r_q = \mathscr{A}^r \otimes_{\mathbf{C}} \Lambda^q\, \mathbf{C}^l$$

and there are maps δ, P which define a double complex

$$\text{(5)} \qquad \begin{array}{ccccccc} & & \downarrow & & \downarrow & & \\ \longrightarrow & \mathscr{A}^r_q & \xrightarrow{\ \delta\ } & \mathscr{A}^{r+1}_q & \longrightarrow & & \\ & \downarrow{\scriptstyle \mathrm{P}} & & \downarrow{\scriptstyle \mathrm{P}} & & & \\ \longrightarrow & \mathscr{A}^r_{q-1} & \xrightarrow{\ \delta\ } & \mathscr{A}^{r+1}_{q-1} & \longrightarrow & & \\ & \downarrow & & \downarrow & & & \end{array}$$

The map δ is the usual co-boundary operator and the contraction map P is defined by the Koszul complex associated to the index q and the functions $\rho_1, \cdots, \rho_l$. Namely, if $\omega = (\omega^{\mathrm{J}}_{\mathrm{I}}) \in \mathscr{A}^r_q$, then

$$(\mathrm{P}\omega)^{\mathrm{J}}_{\mathrm{K}} = \sum_{i=1}^{l} \omega^{\mathrm{J}}_{\mathrm{K}i}\, \rho_i .$$

As in Theorem 5.2 [2] one checks the exactness of (5) with respect to r. To obtain exactness in the other index one can prove the following version of Theorem 5.3 from [2].

THEOREM 4.1. *Let C be a good family with respect to an almost parallel family $\mathscr{L}$. Then*

(i) *For $q \geq 1$, $r \geq 1$, there is a good refinement C' of C such that for all $\omega \in \mathscr{A}^r_q(C)$, $\mathrm{P}\omega = 0$ there is an $\eta \in \mathscr{A}^r_{q+1}(C')$ such that restr $(\omega) = \mathrm{P}(\eta)$, where restr : $\Omega^r_q(C) \to \mathscr{A}^r_q(C')$ is the natural restriction map.*

(ii) *Let $r \geq 0$. There exists a good refinement C' of C such that for all $\omega = (\omega^{\mathrm{J}})$, $\omega \in \mathscr{A}^r_0(C)$ with the property that ω^{J} belongs to the ideal generated by $\rho_1, \cdots, \rho_l$ in the space $\mathrm{H}(\Omega_{j_0} \cap \cdots \cap \Omega_{j_r})$ $(\mathrm{J} = (j_0, \cdots, j_r))$ there exists $\eta \in \Omega^r_1(C')$ such that restr $(\omega) = \mathrm{P}(\eta)$.*

Proof. The proof follows in general lines the one given in [2]. To see the differences it is enough to outline the case $q = r = 0$. Let $\omega \in \Omega^0_0(C)$ and $\Omega \in C$. Then ω determines a function λ analytic on Ω and such that for every $\varepsilon > 0$,

$|\lambda(z)| \leq A_{\varepsilon,k}\, e^{\varepsilon|z|}$ where $A_{\varepsilon,k}$ does not depend on Ω but only on the first integer k such that $\Omega \subset \Gamma(k)$ ($A_{\varepsilon,k}$ is in general a monotone non-decreasing function of k but not necessarily bounded above). Using the hypothesis and interpolation in Ω we obtain, on $\Omega' \subset\subset \Omega$,

$$\lambda = \sum_{j=1}^{l} \alpha_j \rho_j$$

with $|\alpha_j(z)| \leq A' e^{\varepsilon'|z|}$. In particular, as in sections 1 and 2 of [2], we can check that $\forall \varepsilon' > 0\ \exists A'_{\varepsilon,k}$ such that $|\alpha_j(z)| \leq A'_{\varepsilon',k}\, e^{\varepsilon'|z|}$. This yields the thesis.

As a corollary one obtains.

THEOREM 4.2. *If* $\rho\,(\rho_1, \cdots, \rho_m)$ *is slowly decreasing in* Γ *then* $I = I_{loc}$ *and, hence,* I *is closed.*

5. CONVOLUTORS IN $H_c(\Omega(g, \Gamma))$

In section 3 we defined the convolution $\mu * f$ when $f \in H_c(\Omega)$, $\mu \in H'_c(\Omega)$ and $\Omega = \Omega(0, \Gamma)$. The operator $\mu *$ maps $H_c(\Omega)$ into $H_c(\Omega)$ and commutes with translation by the elements in Ω. Since for $g \neq 0$ and $\Omega = \Omega(g, \Gamma)$ we do not have translations at our disposal anymore, one looks for a new way of defining convolutors in $H_c(\Omega)$. Following Ehrenpreis [5] one computes, as in [9], the Fourier transform of the transpose of the operator $\mu *$. It then appears that such a Fourier transform is multiplication by a function m in the space $\mathrm{Exp}_c(0, \Gamma)$. Considering now the case of arbitrary g, we say that a function $f \in H(\Gamma)$ is a *multiplicator* if the map $h \to f \cdot h$ is a continuous map from $\mathrm{Exp}_c(g, \Gamma)$ into $\mathrm{Exp}_c(g, \Gamma)$. For instance, when $g \leq 0$, any element in $\mathrm{Exp}_c(g, \Gamma)$ defines a multiplicator (for $g < 0$, these are not necessarily the only multiplicators). On the other hand, for any g, a function $f \in H(\Gamma)$ of minimal type is a multiplicator in $\mathrm{Exp}_c(g, \Gamma)$. Given a multiplicator m we define a convolutor M in $H_c(\Omega)$ by

$$M := \mathscr{F}^t\, m^t\, (\mathscr{F}^{-1})^t.$$

Note that in the case of distributions in $\mathbf{R}^n$, M would correspond exactly to convolution by $\check{\mu}$ if $m = \hat{\mu}$. We also remark that one could as well define multiplicators from $\mathrm{Exp}_c(g_1, \Gamma)$ into $\mathrm{Exp}_c(g_2, \Gamma)$ giving rise to convolutors from $H_c(\Omega(g_2, \Gamma))$ into $H_c(\Omega(g_1, \Gamma))$.

We will restrict ourselves to convolutors corresponding to functions of minimal type. Consider now a point $p \in \mathbf{C}^n$ and denote by $\mathscr{L}_p$ the family of all complex lines through p. Let $\rho \in H(\Gamma)$ be a function of minimal type, i.e. given $\varepsilon > 0$ there is $A > 0$ such that

$$|\rho(z)| \leq A\, e^{\varepsilon|z|}, \qquad z \in \Gamma,$$

and consider ρ as a multiplicator in $\mathrm{Exp}_c(g, \Gamma)$ (g fixed).

DEFINITION 5.1. We say that ρ is slowly decreasing if there exists a point $p \in \mathbf{C}^n$, an exhaustion $\{\Gamma_k\}$ of Γ and a sequence of positive real numbers $\{\varepsilon_k\}$ such that if $L \cap \Gamma_{k-1} \neq \varnothing$ for some $L \in \mathscr{L}_p$, then, for every $\alpha > 0$, the set

$$S(\rho ; L, \varepsilon_k, \alpha) = \{z \in L \cap \Gamma_k : |\rho(z)| < \varepsilon_k e^{-\alpha|z|}\}$$

has components which are relatively compact and whose diameters are uniformly bounded.

We note how close the above definition is to that of very slowly decreasing distributions introduced by Ehrenpreis [5]. One easily proves the following.

THEOREM 5.1. *Let* ρ *be a slowly decreasing multiplicator in* $\mathrm{Exp}_c(g, \Gamma)$. *Denote by* (ρ) *the set* $\rho \cdot \mathrm{Exp}_c(g, \Gamma)$ *and by* $(\rho)_{\mathrm{loc}}$ *the set of functions in* $\mathrm{Exp}_c(g, \Gamma)$ *which can be written as a product* $\rho \cdot h$, $h \in H(\Gamma)$. *Then*

$$(\rho) = (\rho)_{\mathrm{loc}} .$$

6. INTERPOLATION

Given a slowly decreasing $\rho = (\rho_1, \cdots, \rho_n)$ as considered in section 3 we can study which holomorphic functions on the variety $V = \{\rho_1 = \cdots = \rho_n = 0\}$ can be extended to holomorphic functions in the space $\mathrm{Exp}_c(0, \Gamma)$. Here we consider the variety V with multiplicities, i.e. the space of holomorphic functions on V is $A(V) := H(\Gamma)/\mathscr{T}$, where $\mathscr{T}$ is the closure of the algebraic ideal generated by $\rho_1, \cdots, \rho_n$ in $H(\Gamma)$. We have an associated restriction map

$$\mathrm{restr} : \mathrm{Exp}_c(0, \Gamma) \to A(V)$$

and the interpolation problem consists precisely in characterizing the image of the restriction map. Using the Jacobi interpolation formula (1.5) from [2], we see that given $\lambda \in A(V)$ we have a well defined holomorphic function $I(\lambda)$ on $S(\rho ; k, \varepsilon_k, \alpha_k)$ and require $I(\lambda) \in H\left(\Gamma(k), \exp\left(-\frac{1}{k}|z|\right)\right)$ for each k. For a pair of sequences $\{\varepsilon_k'\}$, $\{\alpha_k'\}$ where $\varepsilon_k' \leq \varepsilon_k$ and $\alpha_k' \geq \alpha_k$ we have a larger class of functions $\lambda \in A(V)$ satisfying these conditions and hence the union of all these classes defines a space $A_G(V)$ "with groupings". One proves without difficulty (compare with Theorem 4.7 from [2]).

THEOREM 6.1. $\mathrm{restr}(\mathrm{Exp}_c(0, \Gamma)) = A_G(V)$.

Suppose that $\mu_j \in H_c'(\Omega)$, $\Omega = \Omega(0, \Gamma)$, $\rho_j = \hat{\mu}_j$, $\rho = (\rho_1, \cdots, \rho_n)$ is slowly decreasing and furthermore, to simplify the notation, all the points in the associated variety V have multiplicity one, $V = \{z_1, z_2, \cdots\} \subset \Gamma$. Then one has the following representation theorem for mean-periodic functions with respect to $\mu_1, \cdots, \mu_n$.

THEOREM 6.2. *Let* $f \in H_c(\Omega)$ *satisfying the system of convolution equations* $\mu_1 * f = \cdots = \mu_n * f = 0$ *then*

$$f(x) = \Sigma c_j \exp \langle x , z_j \rangle$$

where the series is convergent in $H_c(\Omega)$ *after grouping of terms.* (*see* [2], *section 9 and* [11]).

One has similarly an interpolation theorem and corresponding representation theorem for the situation considered in section 4, as was done in [2], [11].

It would be interesting to make explicit the interpolation theorems and the corresponding representation theorems corresponding to section 5 above, even for more general convolutors than the ones introduced. Furthermore, probably even more interesting would be to prove the same kind of theorems for the AU-spaces of Fourier hyperfunctions, ultradistributions, etc given in example 1 of [4], p. 157. In particular, consider the different notions of being slowly decreasing, showing general classes of slowly decreasing elements in the duals of these spaces. As we said in the introduction, the object of the present paper is to indicate how one should proceed in these examples with possible applications to quantum field theory and other problems; we hope we have succeeded in arousing interest in these questions.

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