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On the lower semicontinuity of certain integral functionals


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**Calcolo delle variazioni. — On the lower semicontinuity of certain integral functionals.** Nota di ENNIO DE GIORGI, GIUSEPPE BUTTAZZO e GIANNI DAL MASO (*), presentata (**) dal Corrisp. E. DE GIORGI.

**Riassunto. —** Si dimostra che il funzionale $\int f(u, Du) \, dx$ è semicontinuo inferiormente su $W^{1,1}_{\text{loc}}(\Omega)$, rispetto alla topologia indotta da $L^1_{\text{loc}}(\Omega)$, qualora l'integrando $f(s, p)$ sia una funzione non-negative, misurabile in $s$, convessa in $p$, limitata nell'intorno dei punti del tipo $(s, 0)$, e tale che la funzione $s \mapsto f(s, 0)$ sia semicontinua inferiormente su $\mathbb{R}$.

**Introduction**

Let $n \geq 1$ be an integer and let $\Omega$ be an open subset of $\mathbb{R}^n$. For every $u \in W^{1,1}_{\text{loc}}(\Omega)$ we set $Du = (D_1 u, \ldots, D_n u)$, where $D_i u = \partial u / \partial x_i$. By “measurable” we always mean Lebesgue-measurable. For every $t \in \mathbb{R}$ we set $t^+ = \max \{t, 0\}$. For every function $f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and for every $s \in \mathbb{R}$ we define

$$
\alpha_f(s) = \limsup_{p \to 0} \frac{[f(s, 0) - f(s, p)]^+}{|p|}.
$$

The aim of this paper is to prove the following theorem.

**Theorem 1.** Let $f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a function with the following properties:

(a) for every $s \in \mathbb{R}$ and $p \in \mathbb{R}^n$ we have $f(s, p) \geq 0$;

(b) for every $p \in \mathbb{R}^n$ the function $s \mapsto f(s, p)$ is measurable on $\mathbb{R}$;

(c) for every $s \in \mathbb{R}$ the function $p \mapsto f(s, p)$ is convex on $\mathbb{R}^n$;

(d) the function $s \mapsto f(s, 0)$ is lower semicontinuous on $\mathbb{R}$;

(e) the function $\alpha_f$ belongs to $L^1_{\text{loc}}(\mathbb{R})$.

Then for every $u \in W^{1,1}_{\text{loc}}(\Omega)$ the function $x \mapsto f(u(x), Du(x))$ is measurable and the functional $F: W^{1,1}_{\text{loc}}(\Omega) \to [0, +\infty]$ defined by

$$
F(u) = \int_\Omega f(u, Du) \, dx
$$

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is lower semicontinuous on $W_{\text{loc}}^{1,1}(\Omega)$ with respect to the topology induced by $L_{\text{loc}}^1(\Omega)$.

**Remark 1.** This theorem differs from other semicontinuity results (see [2], [5], [7] Chapter 4, [8] [9]) chiefly in that we do not assume that the function $s \mapsto f(s, p)$ is continuous or lower semicontinuous, except for $p = 0$. This allows us to include in a general framework the case of functionals of the form

$$
\int_\Omega \left( \sum_{i,j=1}^n a_{i,j}(u) D_i u D_j u \right)^q \, dx
$$

where $q \geq 1/2$ and $a_{i,j}$ are measurable functions such that

$$
\sum_{i,j=1}^n a_{i,j}(s) p_i p_j \geq 0 \quad \text{for every } s \in \mathbb{R}, p \in \mathbb{R}^n.
$$

**Remark 2.** If $f$ satisfies conditions (a), (b), (c) of Theorem 1, then condition (e) is satisfied whenever there exist $\varepsilon > 0$ and $\beta \in L_{\text{loc}}^1(\mathbb{R})$ such that $f(s, p) \leq \beta(s)$ for every $s \in \mathbb{R}$ and for every $p \in \mathbb{R}^n$ with $|p| \leq \varepsilon$.

**Remark 3.** Hypothesis (e) in Theorem 1 cannot be dropped, as the following example shows. Let $n = 1$, $\Omega = [0, 1]$, and let $f$ be defined by

$$
f(s, p) = \begin{cases} 
1 + \frac{p}{s} & \text{if } s \neq 0 \\
1 & \text{if } s = 0.
\end{cases}
$$

For every $\varepsilon > 0$ let $u_\varepsilon(x) = \varepsilon - \varepsilon x$. Then $(u_\varepsilon)$ converges to 0 as $\varepsilon \to 0$, but $F(u_\varepsilon) = 0$, whereas $F(0) = 1$. Note that $f$ satisfies all conditions of Theorem 1 except (e).

**Preliminary Lemmas.**

For every $x, y \in \mathbb{R}^n$ we denote by $\langle x, y \rangle$ the scalar product of $x$ and $y$ and by $|x|$ the Euclidean norm of $x$.

**Lemma 1.** Let $u \in W_{\text{loc}}^{1,1}(\Omega)$ and let $E$ be a Borel subset of $\mathbb{R}$ with $\text{meas}(E) = 0$. Then $Du = 0$ a.e. on $u^{-1}(E)$.

**Proof.** The proof follows easily from a result of De La Vallée Poussin (see [3], [10]).

**Definition 1.** We say that a function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is an integrand if:

(a) for every $p \in \mathbb{R}^n$ the function $s \mapsto f(s, p)$ is measurable on $\mathbb{R}$;

(b) for every $s \in \mathbb{R}$ the function $p \mapsto f(s, p)$ is continuous on $\mathbb{R}$;

(c) the function $s \mapsto f(s, 0)$ is a Borel function.
DEFINITION 2. We say that two integrands \( f, g \) are equivalent integrands if there exists a Borel set \( N \subseteq \mathbb{R} \) with \( \text{meas}(N) = 0 \) such that

(a) for every \( s \in \mathbb{R} - N \) and \( p \in \mathbb{R}^n \) we have \( f(s, p) = g(s, p) \);

(b) for every \( s \in \mathbb{R} \) we have \( f(s, 0) = g(s, 0) \).

LEMMA 2. If \( f, g \) are equivalent integrands and \( u \in W^{1,1}_{\text{loc}}(\Omega) \), then \( f(u(x), Du(x)) = g(u(x), Du(x)) \) a.e. on \( \Omega \).

Proof. It follows from Lemma 1.

LEMMA 3. If \( f \) is an integrand and \( u \in W^{1,1}_{\text{loc}}(\Omega) \), then the function \( x \mapsto f(u(x), Du(x)) \) is measurable on \( \Omega \).

Proof. There exists a Borel function \( g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) such that \( f \) and \( g \) are equivalent integrands. The result now follows from Lemma 2.

LEMMA 4. Let \( a : \mathbb{R} \to \mathbb{R} \) be a Lipschitz continuous function and let \( b : \mathbb{R} \to \mathbb{R} \) be a bounded measurable function such that \( a'(s) = b(s) \) a.e. on \( \mathbb{R} \). If \( u \in W^{1,1}_{\text{loc}}(\Omega) \) and \( v = a \circ u \), then \( v \in W^{1,1}_{\text{loc}}(\Omega) \) and \( Du = b(u) Du \) a.e. on \( \Omega \).

Proof. See [6] Lemma 1.2 and Lemma 1.5.

LEMMA 5. Let \( b \in L^1(\mathbb{R}, \mathbb{R}^n) \) and let \( a : \mathbb{R} \to \mathbb{R}^n \) be defined by \( a(t) = \int_0^t b(s) \, ds \). Let \( u \in W^{1,1}_{\text{loc}}(\Omega) \) be a function such that

\[ \int_\Omega |b(u) \cdot Du|^+ \, dx < +\infty. \]

Then, for every \( \varphi \in C_0^\infty(\Omega) \) with \( \varphi \geq 0 \), the function \( \langle b(u), Du \rangle \varphi \) is in \( L^1(\Omega) \) and

\[ \int_\Omega \langle b(u), Du \rangle \varphi \, dx = -\int_\Omega \langle a(u), D\varphi \rangle \, dx. \]

Proof. If \( b \) is bounded, the thesis follows from Lemma 4. In the general case it suffices to approximate \( b \) by the sequence \( (b_n) \) defined by

\[ b_n(s) = \begin{cases} b(s) & \text{if } |b(s)| \leq h \\ 0 & \text{otherwise} \end{cases} \]
Lemma 6. Let \((f_n)\) be a sequence of non-negative measurable functions from \(\mathbb{R}^n\) into \(\mathbb{R}\) and let \(f_\infty = \sup_h f_h\). Then for every open subset \(A\) of \(\mathbb{R}^n\) we have
\[
\int_A f_\infty (x) \, dx = \sup_{k \in \mathbb{N}} \sup \left\{ \sum_{i=1}^k \int_{A_i} f_i (x) \, dx : A_1, \ldots, A_k \text{ pairwise disjoint open subsets of } A \right\}.
\]

Proof. For every \(k \in \mathbb{N}\) set \(g_k = \sup \{f_i : i = 1, \ldots, k\}\); then by Beppo Levi's theorem we have
\[
\int_A f_\infty (x) \, dx = \sup_{k \in \mathbb{N}} \int_A g_k (x) \, dx.
\]
Now fix \(k \in \mathbb{N}\); there exist measurable pairwise disjoint subsets \(B_1, \ldots, B_k\) of \(A\) such that \(g_k = f_i\) on \(B_i\). Then
\[
\int_A g_k (x) \, dx = \sum_{i=1}^k \int_{B_i} f_i (x) \, dx = \sup \left\{ \sum_{i=1}^k \int_{K_i} f_i (x) \, dx : K_i \subseteq B_i ; K_i \text{ compact} \right\} = \sup \left\{ \sum_{i=1}^k \int_{A_i} f_i (x) \, dx : A_1, \ldots, A_k \text{ pairwise disjoint open subsets of } A \right\}.
\]

Lemma 7. Let \((f_h)\) be a sequence of non-negative integrands and let \(f_\infty = \sup_h f_h\).
Set for every open subset \(A\) of \(\Omega\), every \(u \in W^{1,1}_{\text{loc}} (A)\), and every \(h \in \mathbb{N} \cup \{\infty\}\)
\[
F_h (u, A) = \int_A f_h (u, Du) \, dx.
\]
Suppose that for every \(h \in \mathbb{N}\) and every open subset \(A\) of \(\Omega\) the functional \(F_h (\cdot, A)\) is \(L^1_{\text{loc}} (A)\)-lower semicontinuous. Then, for every open subset \(A\) of \(\Omega\) the functional \(F_\infty (\cdot, A)\) is \(L^1_{\text{loc}} (A)\)-lower semicontinuous.

Proof. It follows from Lemma 6.

Proof of Theorem 1.

The proof of Theorem 1 is divided into two parts. In the first one we deal with the case \(f(s, 0) = 0\) (considered in Lemma 10); then we shall use this partial result to prove the general case. The measurability of the function \(x \mapsto f(u(x), Du(x))\) has already been proved in Lemma 3.
The functionals we are going to consider are defined in $W^{1}_{loc}(\Omega)$; when we say that a functional $F$ is lower semicontinuous, we mean that $F$ is lower semicontinuous on $W^{1}_{loc}(\Omega)$ with respect to the topology induced by $L^{1}_{loc}(\Omega)$. For every $B \subseteq \mathbb{R}$ we indicate by $1_{B}$ the characteristic function of $B$, defined by $1_{B}(s) = 1$ if $s \in B$ and $1_{B}(s) = 0$ if $s \in \mathbb{R} - B$.

**Lemma 8.** Let $b : \mathbb{R} \to \mathbb{R}^{n}$ be a measurable function and let $g : \mathbb{R} \to \mathbb{R}$ be a lower semicontinuous function with $g \leq 0$. Then the functional

$$F(u) = \int_{\Omega} \left[ g(u) + \langle b(u), Du \rangle \right]^{+} dx$$

is lower semicontinuous.

**Proof.** First assume that $b$ and $g$ are bounded. For every $u \in W^{1,1}_{loc}(\Omega)$ we have

$$F(u) = \sup \left\{ \int_{\Omega} \left[ g(u) + \langle b(u), Du \rangle \right] \varphi \, dx : \varphi \in C_{0}^{\infty}(\Omega), 0 \leq \varphi \leq 1 \right\};$$

therefore it is enough to prove that for every $\varphi \in C_{0}^{\infty}(\Omega)$, with $\varphi \geq 0$, the functionals

$$G(u) = \int_{\Omega} g(u) \varphi \, dx$$

$$H(u) = \int_{\Omega} \langle b(u), Du \rangle \varphi \, dx$$

are lower semicontinuous. For $G$ it is enough to apply Fatou’s lemma. From Lemma 4 we obtain

$$H(u) = \int_{\Omega} \text{div}(a \circ u) \varphi \, dx = - \int_{\Omega} \langle a(u), D\varphi \rangle \, dx$$

where $a(t) = \int_{0}^{t} b(s) \, ds$.

This implies that $H$ is continuous on $W^{1,1}_{loc}(\Omega)$ with respect to the topology induced by $L^{1}_{loc}(\Omega)$.

If $b$ or $g$ are unbounded, let $(b_{h})$ be the sequence of functions defined by

$$b_{h}(s) = \begin{cases} b(s) & \text{if } |b(s)| \leq h \\ 0 & \text{otherwise} \end{cases}$$
and let \((\sigma_h)\) be an increasing sequence of functions in \(C_0^\infty(\mathbb{R})\) with \(\sigma_h \geq 0\) and 
\[ \lim_h \sigma_h(s) = 1 \quad \text{for every } s \in \mathbb{R}. \]
Since \(g\) is lower semicontinuous and \(g \leq 0\), every function \(\sigma_h(s)g(s)\) is bounded. By Beppo Levi's theorem
\[
F(u) = \sup_{h \in \mathbb{N}} \int_{\Omega} \left[ \sigma_h(u)g(u) + (\sigma_h(u)b(u), Du) \right]^+ dx.
\]
Therefore the lower semicontinuity of \(F\) follows from the result obtained in the bounded case.

**Lemma 9.** Let \(b: \mathbb{R} \to \mathbb{R}^n\) be a measurable function and let \(g: \mathbb{R} \to \mathbb{R}\) be a measurable function with \(g \leq 0\). Then the functional
\[
F(u) = \int_{\Omega} \left[ g(u) + (b(u), Du) \right]^+ dx
\]
is lower semicontinuous.

**Proof.** By Lusin's theorem there exists an increasing sequence \((K_h)\) of compact subsets of \(\mathbb{R}\) and a sequence \((g_h)\) of continuous functions with \(g_h \leq 0\), such that \(g_h(s) = g(s)\) for every \(s \in K_h\) and \(\text{meas}(\mathbb{R} - E) = 0\), where \(E = \bigcup_h K_h\).
Since \(g \leq 0\), using Lemma 2 and Beppo Levi's Theorem, we get
\[
F(u) = \int_{\Omega} 1_{E}(u) \left[ g(u) + (b(u), Du) \right]^+ dx =
\]
\[
= \sup_{h \in \mathbb{N}} \int_{\Omega} \left[ 1_{K_h}(u) g_h(u) + (1_{K_h}(u)b(u), Du) \right]^+ dx
\]
for every \(u \in W_{loc}^{1,1}(\Omega)\). Since \(g_h \leq 0\), the functions \(1_{K_h}(s)g_h(s)\) are lower semicontinuous, thus the lower semicontinuity of \(F\) follows from Lemma 8.

**Lemma 10.** Assume that \(f\) satisfies conditions (a), (b) (c) of Theorem 1, and that \(f(s, 0) = 0\) for every \(s \in \mathbb{R}\). Then the functional
\[
F(u) = \int_{\Omega} f(u, Du) dx
\]
is lower semicontinuous.

**Proof.** For every \(s \in \mathbb{R}\) set
\[
K(s) = \{(a, b) \in \mathbb{R} \times \mathbb{R}^n : f(s, p) \geq a + |b|, \forall p \in \mathbb{R}^n\}.
\]
By the measurable selection theorem (see [1] Th. III, 30 page 80) there exist a sequence \((a_h)\) of measurable functions from \(\mathbb{R}\) into \(\mathbb{R}\), and a sequence \((b_h)\)
of measurable functions from $\mathbb{R}$ into $\mathbb{R}^n$, such that for every $s \in \mathbb{R}$ the set \{(a_h(s), b_h(s)) : h \in \mathbb{N}\} is dense in $K(s)$. Then for every $s \in \mathbb{R}$, $p \in \mathbb{R}^n$

(1) $f(s, p) = \sup \{ [a + \{b, p\}^+ : (a, b) \in K(s) \} = \sup_{h \in \mathbb{N}} \{a_h(s) + \{b_h(s), p\}^+\}$

Since $f(s, 0) = 0$, by (1) we have $a_h(s) \leq 0$. Thus the lower semicontinuity of $F$ follows from Lemma 9 and from Lemma 7.

**Lemma 11.** Let $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$. Under the assumptions of Lemma 10 the functional

$$F(u) = \int_\Omega f(u, Du) \varphi \, dx$$

is lower semicontinuous.

**Proof.** For every $h, k \in \mathbb{N}$ let $\Omega_{h,k} = \{x \in \Omega : \varphi(x) > k 2^{-h}\}$ and let

$$\varphi_h(x) = 2^{-h} \sum_{k=1}^{4^h} 1_{\Omega_{h,k}}(x).$$

The sequence $(\varphi_h)$ is increasing and $\varphi = \sup_{h \in \mathbb{N}} \varphi_h$. Then

$$F(u) = \sup_{h \in \mathbb{N}} \int_\Omega f(u, Du) \varphi_h \, dx = \sup_{h \in \mathbb{N}} 2^{-h} \sum_{k=1}^{4^h} \int_{\Omega_{h,k}} f(u, Du) \, dx.$$ 

Thus the lower semicontinuity of $F$ follows from Lemma 10.

**Proof of Theorem 1.** Assume first that $\alpha_f \in L^1(\mathbb{R})$. For every $s \in \mathbb{R}$ let $\partial f(s, 0)$ be the subdifferential at the point $p = 0$ of the convex function $p \mapsto f(s, p)$ and let $b(s)$ be the element of $\partial f(s, 0)$ such that $|b(s)| = \min \{|q| : q \in \partial f(s, 0)\}$.

It is known that $b : \mathbb{R} \to \mathbb{R}^n$ is measurable (see [4], Th. 1.2, page 236) and that $|b(s)| = \alpha_f(s)$ for every $s \in \mathbb{R}$. Since

(2) $f(s, p) \geq f(s, 0) + \{b(s), p\}$

for every $s \in \mathbb{R}$, $p \in \mathbb{R}^n$, the function

(3) $g(s, p) = f(s, p) - f(s, 0) - \{b(s), p\}$

satisfies all conditions of Lemma 10.

Let $(u_h)$ be a sequence in $W^{1,1}_{\text{loc}}(\Omega)$ converging in $L^1_{\text{loc}}(\Omega)$ to a function $u_\infty \in W^{1,1}_{\text{loc}}(\Omega)$; we have to prove that

(4) $F(u_\infty) \leq \liminf_h F(u_h)$. 

If the right-hand side is $-\infty$ the inequality is trivial. So we may assume that 
\[ \liminf_{h \to \infty} F(u_h) < +\infty \] and that $F(u_h) < +\infty$ for every $h \in \mathbb{N}$. Since $f(s, p) \geq 0$ by (2) we obtain
\[ \int_{\Omega} \langle b(u_h), Du \rangle^+ \, dx \leq F(u_h) < +\infty. \]

Since the function $\langle b(s), p \rangle^+$ satisfies all conditions of Lemma 10 we have
\[ \int_{\Omega} \langle b(u_{\infty}), Du_{\infty} \rangle^+ \, dx \leq \liminf_{h \to \infty} \int_{\Omega} \langle b(u_h), Du_h \rangle^+ \, dx \leq \liminf_{h \to \infty} F(u_h) < +\infty. \]

Let $\varphi \in \mathcal{C}^\infty_0(\Omega)$ with $0 \leq \varphi \leq 1$. For every $s \in \mathbb{R}$ set
\[ a(t) = \int_{0}^{t} b(s) \, ds; \]
by Lemma 5
\[ \int_{\Omega} \langle b(u_h), Du_h \rangle \varphi \, dx = - \int_{\Omega} \langle a(u_h), D\varphi \rangle \, dx \] for every $h \in \mathbb{N} \cup \{\infty\}$. By Lemma 11
\[ \int_{\Omega} g(u_{\infty}, Du_{\infty}) \varphi \, dx \leq \liminf_{h \to \infty} \int_{\Omega} g(u_h, Du_h) \varphi \, dx. \]
Since the function $s \mapsto f(s, 0)$ is lower semicontinuous, by Fatou's Lemma
\[ \int_{\Omega} f(u_{\infty}, 0) \varphi \, dx \leq \liminf_{h \to \infty} \int_{\Omega} f(u_h, 0) \varphi \, dx. \]

Since $a$ is continuous and bounded, from (5) we get
\[ \int_{\Omega} \langle b(u_{\infty}), Du_{\infty} \rangle \varphi \, dx = \lim_{h \to \infty} \int_{\Omega} \langle b(u_h), Du_h \rangle \varphi \, dx. \]

From (3), (6), (7), (8) we obtain
\[ \int_{\Omega} f(u_{\infty}, Du_{\infty}) \varphi \, dx \leq \liminf_{h \to \infty} \int_{\Omega} f(u_h, Du_h) \varphi \, dx \leq \liminf_{h \to \infty} F(u_h). \]
Since

\[ F(u) = \sup \left\{ \int_{\Omega} f(u, Du) \varphi \; dx : \varphi \in C^\infty_0(\Omega), \; 0 \leq \varphi \leq 1 \right\} \]

we get (4) and the Theorem is proved in the case \( \alpha_f \in L^1(\mathbb{R}) \).

In the general case \( \alpha_f \in L^{1,\text{loc}}(\mathbb{R}) \), let \((\sigma_h)\) be an increasing sequence of functions of \( C^\infty_0(\mathbb{R}) \) with \( \sigma_h \geq 0 \) and \( \lim_{h} \sigma_h(s) = 1 \) for every \( s \in \mathbb{R} \), let

\[ f_h(s,p) = \sigma_h(s)f(s,p) \]

for every \( s \in \mathbb{R}, \; p \in \mathbb{R}^n \), and let

\[ F_h(u) = \int_{\Omega} f_h(u, Du) \; dx \]

For every \( u \in W^{1,1}_{\text{loc}}(\Omega) \) we have

\[ F(u) = \sup_{h} F_h(u) \]

Since \( \alpha_{f_h} \in L^1(\mathbb{R}) \) the functionals \( F_h \) are lower semicontinuous; hence \( F \) is lower semicontinuous and the Theorem is proved.

**References**


