Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **74** (1983), n.5, p. 267–273.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1983_8_74_5_267_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1983.

RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta del 14 maggio 1983

Presiede il Presidente della Classe GIUSEPPE MONTALENTI

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Analisi matematica. — *Interpolation problems in cones.* Nota I di CARLOS A. BERENSTEIN e DANIELE STRUPPA, presentata (*) dal Corrisp. E. VESENTINI.

RIASSUNTO. — In questa nota, si studiano problemi di interpolazione per varietà discrete in spazi di funzioni olomorfe in coni.

In particolare si mostra come sia possibile estendere il Principio Fondamentale di Ehrenpreis ad equazioni di convoluzione nella spazio $H_c(\Omega)$, introdotto in [4] in connessione con problemi di fisica quantistica.

INTRODUCTION

It was shown by Ehrenpreis and Palamodov [5], [10] that the class of AU-spaces is specially suited to the study of partial differential equations with constant coefficients (PDE). The examples of AU-spaces given in [5], [10] and [1] are, loosely speaking, related to spaces of distributions of compact support. In this way the analysis of systems of PDE is reduced to interpolation problems in spaces of entire functions in $\mathbf{C}^n$ with growth conditions.

For instance, for the AU-space $\mathscr{E}(\mathbf{R}^n)$ of C^∞-functions on $\mathbf{R}^n$, one studies interpolation problems in the space $\hat{\mathscr{E}}'(\mathbf{R}^n)$ of Fourier transforms of distributions with compact support in $\mathbf{R}^n$. The Paley-Wiener theorem describes $\hat{\mathscr{E}}'(\mathbf{R}^n)$ as the space

$$\hat{\mathscr{E}}'(\mathbf{R}^n) = \{f \text{ entire function in } \mathbf{C}^n : \exists\, A, B > 0 \text{ such that}$$
$$|f(z)| \leq A e^{Bp(z)} \quad \text{for all } z \in \mathbf{C}^n\}$$

(*) Nella seduta del 23 aprile 1983.

where $p(z) = |\operatorname{Im} z| + \log(2 + |z|)$. Furthermore, the space $\hat{\mathcal{E}}'(\mathbf{R}^n)$ turns out to be an algebra so that the interpolation problems are tied to the ideal theory in such algebra.

It is via this route that Ehrenpreis proved his Fundamental Principle for PDE. This principle states that every solution of an homogeneous system of PDE can be represented in terms of the exponential solutions of the same system.

In [2] and [11] the Fundamental Principle was extended to convolution equations in the same kind of AU-spaces considered by Ehrenpreis and Palamodov. It turns out though, that a number of spaces of interest in quantum field theory, for instance, correspond to spaces of analytic functionals with unbounded carriers. In view of their possible physical applications, de Roever [4] shows that a number of these spaces could also be considered AU-spaces and proved that the Fundamental Principle for PDE was still valid. Clearly the same kind of physical applications shows that one should also study delay-differential equations and other integro-differential equations of the convolution type. The new problem that arises in the spaces considered in [4] is that one is forced to study interpolation problems in spaces of holomorphic functions in cones, and furthermore these spaces are often not algebras. The aim of this paper is to show how one can extend the work from [2], [11] to the AU-space $H_c(\Omega)$ considered by de Roever. This space should serve as a model on how to extend the Fundamental Principle to the whole list from [4]. This space $H_c(\Omega)$, to be described precisely in the next section, is a space of functions holomorphic in a convex set Ω of $\mathbf{C}^n$. The recent work of Meril [9] in the same direction considered the space $H_c(\Omega)$ only when $n = 1$ and Ω is a cone; restrictions which we do not need to impose.

The first author would like to thank the National Science Foundation and General Research Board of the University of Maryland for their generous support.

2. Basic definitions

In this section we recall from [4] the definition of the space $H_c(\Omega)$ which is the one we will deal with throughout this paper. Let Ω be an open connected region of $\mathbf{C}^n$, denote by $H(\Omega)$ the ring of holomorphic functions on Ω. If M is a weight on Ω, i.e. a positive continuous function on Ω, let $H(\Omega, M)$ be the normed space defined by

$$H(\Omega, M) := \{f \in H(\Omega) : \sup_{z \in \Omega} |f(z)| M(z) < \infty\}.$$

In the following, Γ will always denote open convex cones in $\mathbf{C}^n$ with vertex at the origin. If Γ is such a cone and S^{2n-1} denotes the unit sphere we shall write

$$pr(\Gamma) = \Gamma \cap S^{2n-1}.$$

The following is a well-known lemma.

LEMMA 2.1 (see [4]). *Every closed set* $\Omega \subseteq \mathbf{C}^n$ *which does not contain any real line determines an open convex cone* Γ *and a convex, homogeneous of degree one function* g *on* Γ *such that*

$$\Omega = \Omega(g, \Gamma) := \{\zeta \in \mathbf{C}^n : -\operatorname{Im}\langle y, \zeta\rangle \leq g(y)\ \forall y \in \Gamma\},$$

where $\langle y, \zeta\rangle = y_1 \zeta_1 + \cdots + y_n \zeta_n$. *Conversely, every pair* (g, Γ) *as above determines a closed convex set* $\Omega(g, \Gamma)$.

We shall say that Ω and (g, Γ) are in *duality*.

In [4] de Roever generalized to these convex sets $\Omega(g, \Gamma)$ the Ehrenpreis-Martineau theorem [5], [8] which dealt with the space of holomorphic functions in a bounded convex set Ω, i.e. $\Gamma = \mathbf{C}^n$. We recall briefly here de Roever's result.

If Γ is as before, a subcone Γ' of Γ is said to be a *relatively compact cone* ($\Gamma' \subset\subset \Gamma$) if $pr(\Gamma')$ is a relatively compact subset of $pr(\Gamma)$. We also say that a family $\{\Gamma_k\}_{k\geq 1}$ of open convex cones is an *exhaustion of* Γ if:

$$\Gamma_k \subset\subset \Gamma_{k+1} \subset\subset \Gamma \qquad \text{and} \quad \Gamma = \bigcup_{k\geq 1} \Gamma_k .$$

Given a pair (g, Γ) and an exhaustion $\{\Gamma_k\}$ of Γ we set

$$\Omega_{k,c} := \Omega\left(g(z) + \frac{1}{k}|z|, \Gamma_k\right)$$

and let $\mathring{\Omega}_{k,c}$ its interior. One can now define the space $H_c(\Omega)$ of holomorphic functions in Ω.

DEFINITION 2.1. $H_c(\Omega) := \varinjlim_{k\geq 1} H\left(\mathring{\Omega}_{k,c}, \exp\frac{|z|}{k}\right)$.

(It is easily seen that this definition is independent of the choice $\{\Gamma_k\}$ of exhaustion of Γ).

The set up of the Ehrenpreis-Martineau theorem requires us to define the space of functions of exponential type. Let $\Gamma(k)$ denote the convex set

$$\Gamma(k) := \Gamma_k \cap \left\{z \in \mathbf{C}^n : |z| > \frac{1}{k}\right\}.$$

DEFINITION 2.2. $\operatorname{Exp}_c(g, \Gamma) := \varprojlim_{k\geq 1} H\left(\Gamma(k), \exp\left(-g(z) - \frac{1}{k}|z|\right)\right)$.

It is not hard to see that $\operatorname{Exp}_c(g, \Gamma)$ could also be defined as the space of functions f holomorphic in Γ such that for any $\Gamma' \subset\subset \Gamma$ and any $\varepsilon, \delta > 0$ one has

$$\sup_{\substack{z \in \Gamma' \\ |z| > \delta}} |f(z) \exp(-g(z) - \varepsilon|z|)| < \infty .$$

Finally, if $\zeta \in \Gamma$, one sees that the function

$$z \to e^{i\langle z, \zeta\rangle}$$

is an element of $H_c(\Omega)$, so that, for $\mu \in H_c'(\Omega)$ we can define its *Fourier transform* $\mathscr{F}\mu$

$$\mathscr{F}\mu(\zeta) = \hat{\mu}(\zeta) := \langle \mu_z , e^{i\langle z , \zeta\rangle} \rangle ,$$

where μ_z indicates that μ operates in the variable z. The following result, which generalizes the Ehrenpreis-Martineau theorem, holds (see [4] for its proof.).

THEOREM 2.1. *The Fourier transform $\mathscr{F}$ establishes a topological isomorphism between* $H_c'(\Omega)$ *and* $\mathrm{Exp}_c(g , \Gamma)$.

3. Ideals in $\mathrm{Exp}_c(0 , \Gamma)$. Discrete case

The study of convolution equations in $H_c(\Omega)$ becomes easier when $\mathrm{Exp}_c(g , \Gamma)$ is a commutative algebra. This is precisely the case when $\Omega = \Omega(0 , \Gamma)$, i.e. $g \equiv 0$. Note that in this case Ω is a convex cone with vertex at the origin. This was precisely the case considered for $n = 1$, by Meril in [9]. We postpone the discussion of the latter case ($g \not\equiv 0$) to a later section.

Since Ω is a convex cone, if $x , y \in \Omega$ then also $x + y \in \Omega$. Hence, one easily checks that for $f \in H_c(\Omega)$, $\zeta \in \Omega$ we also have

$$z \to f(z + \zeta) \in H_c(\Omega) .$$

Therefore, one can define, for $\mu \in H_c'(\Omega)$

$$\mu * f(\zeta) := \langle \mu_z , f(z + \zeta) \rangle ,$$

which is an element of $H_c(\Omega)$. As usual, we say that f is mean-periodic (with respect to $\mu \neq 0$) if $\mu * f = 0$. From the general approach indicated in the introduction we know that the study of mean-periodic functions reduces to the study of ideals in the space $\mathrm{Exp}_c(0 , \Gamma)$.

Let us start with the case of one variable. Let $\rho = (\rho_1 , \cdots , \rho_l)$ be an l-tuple of functions in $\mathrm{Exp}_c(0 , \Gamma)$ and consider the ideal $I = I(\rho)$ generated by the ρ_j. Its multiplicity variety $V = V(I)$ is the set of pairs $(z , m) \in \Gamma \times \mathbf{N}$ such that $\rho_j(z) = 0$ for $j = 1 , \cdots , l$ and m is the minimum of the orders of the ρ_j at z. Recall that the *local ideal* $I_{loc}(\rho)$ defined by the ρ_j is the set of all functions $g \in \mathrm{Exp}_c(0 , \Gamma)$ such that for every point $z \in \Gamma$ there is an open neighborhood U and functions $g_1 , \cdots , g_l \in H(U)$ satisfying

$$g = g_1 \rho_1 + \cdots + g_l \rho_l \qquad \text{in} \quad U .$$

It turns out that $I_{loc}(\rho)$ is a closed ideal in $\mathrm{Exp}_c(0 , \Gamma)$ (see [6], [2]). Let $\{\Gamma_k\}_{k \geq 1}$ be an exhaustion of Γ. Given two sequences $\{\varepsilon_k\}$, $\{\alpha_k\}$ or real numbers $\varepsilon_k > 0$, $\alpha_k \geq 0$, define

$$S(\rho ; k , \varepsilon_k , \alpha_k) := \{z \in \Gamma(k) : |\rho(z)| < \varepsilon_k e^{-\alpha_k |z|}\} ,$$

where, as usual,

$$|\rho(z)| = \left(\sum_1^l |\rho_j(z)|^2\right)^{\frac{1}{2}}.$$

DEFINITION 3.1. A function $\rho \in \mathrm{Exp}_c(0, \Gamma)$ (or a vector valued function $\rho = (\rho_1, \cdots, \rho_l)$) is said to be slowly decreasing if there exist an exhaustion $\{\Gamma_k\}$ of Γ and two sequences $\{\varepsilon_k\}, \{\alpha_k\}$ of real numbers ($\varepsilon_k > 0$, $\alpha_k \geq 0$) with $\alpha_k \searrow 0$, such that the connected components of the sets $S(\rho; k, \varepsilon_k, \alpha_k)$ are relatively compact in $\Gamma(k)$ and have uniformly bounded diameters (for each k).

We observe that this is the natural extension of the definition in [2]. Also note that the definition given in [9] assumes $\alpha_k \equiv 0$.

One can prove now the following result.

THEOREM 3.1 ($n = 1$). *If* $\rho \in \mathrm{Exp}_c(0, \Gamma)$ *is a slowly decreasing function of one complex variable then* $I(\rho) = I_{loc}(\rho)$.

Proof. Let $g \in H(\Gamma)$ such that $g\rho \in \mathrm{Exp}_c(0, \Gamma)$. It is enough to show that $g \in \mathrm{Exp}_c(0, \Gamma)$ (this is clearly a division theorem, the reader is referred to [2] for a similar situation). Hence we have that for every $\varepsilon > 0$ and every $k \geq 1$ there is a positive constant $A = A(k)$ such that

$$(1) \qquad |g(z)\rho(z)| \leq Ae^{\varepsilon|z|} \quad \text{on} \quad \Gamma(k),$$

and our aim is to show that for every $\delta > 0$ and every $k \geq 1$ there is $B > 0$ such that

$$(2) \qquad |g(z)| \leq Be^{\delta|z|} \quad \text{on} \quad \Gamma(k).$$

It is clearly enough to show that (2) holds for all $k \geq k_0 = k_0(\delta) \geq 1$. Since $\alpha_k \to 0$ we can find k_0 such that $\varepsilon = \delta - \alpha_{k_0} > 0$. On the boundary ∂S of $S = S(\rho; k, \varepsilon_k, \alpha_k)$ we have $|\rho(z)| = \varepsilon_k e^{-\alpha_k|z|}$. Hence on ∂S,

$$(3) \qquad |g(z)| \leq \frac{A}{\varepsilon_k} e^{(\alpha_k+\varepsilon)|z|} \leq \frac{A}{\varepsilon_k} e^{\delta|z|}.$$

Since the components of S have uniformly bounded diameter the function $|z|$ remains essentially constant on each component and we can apply the maximum principle to g, obtaining (2) in $\bar{S}$ with $B = B(k, \delta) \geq A/\varepsilon_k$. Clearly on $\Gamma(k) \setminus \bar{S}$ the same inequality is immediate. □

Using the solvability of the non-homogeneous $\bar{\partial}$-equation with bounds ([6]) one can also show that if $\rho = (\rho_1, \cdots, \rho_l)$ is slowly decreasing, then $I = I_{loc}$ (see [2], [7] or the proof to Theorem 3.3 below).

We examine now the case of a discrete variety in $\mathbf{C}^n$, $n > 1$. Definition 3.1 makes sense for a vector valued function $\rho = (\rho_1, \cdots, \rho_l)$ and it implies $\rho^{-1}(0)$ is a discrete variety V. We furthermore will assume that V is a complete intersection, that is $l = n$. Before stating the analogous theorem to Theorem 3.1 for $n > 1$ we recall the following result ([4], Theorem 7.3, see also [9], chapter 3) which extends a well-known result from [6].

THEOREM 3.2. *Let* $\{\Gamma_k\}$ *be an exhaustion of the cone* Γ, δ_k *a decreasing sequence of positive numbers and* Φ_k *the plurisubharmonic functions in* Γ, $\Phi_k(z) = \delta_k |z|$. *For any sequence of positive numbers* $\{K_k\}$ *there is a sequence* $\{M_k\}$ *such that for every* $(0, q+1)$ *form* g *with coefficients in* $L^2_{loc}(\Gamma)$ *and* $\bar\partial g = 0$, *there exists a* $(0, q)$ *form* u *with coefficients in* $L^2_{loc}(\Gamma)$ *satisfying* $\bar\partial u = g$, *and, for every* k, *we have*

$$\int_{\Gamma(k)} |g(z)|^2 e^{-2\Phi_k(z)} |dz| \leq K_k$$

implies

$$\int_{\Gamma(k)} |u(z)|^2 \frac{e^{-2\Phi_k(z)}}{(1+|z|)^2} |dz| \leq M_k .$$

where $|dz|$ *denotes the Lebesgue measure in* $\mathbf{C}^n$.

We show now that Theorem 3.1 can be extended to the case of n complex variables.

THEOREM 3.3. *If* $\rho = (\rho_1, \cdots, \rho_n)$ *is slowly decreasing in* $\mathrm{Exp}_c(0 . \Gamma)$, $\Gamma \subseteq \mathbf{C}^n$, *then* $I(\rho) = I_{loc}(\rho)$.

Proof. We follow closely the proof of Theorem 4.2 in [2]. Start by choosing $\Gamma_k, \varepsilon_k, \alpha_k$ so that one can apply the interpolation results from [2] to all the components of $S(\rho; k, \varepsilon_k, \alpha_k)$ for every k. This can be done since ρ is slowly decreasing. Consider now a function $\lambda \in I_{loc}$, we want to prove $\lambda \in I$. Since $\lambda \in I_{loc}$ and Γ is convex one can apply Cartan's Theorem B [6] to otbain

$$\lambda = \sum_{j=1}^{n} f_j \rho_j \quad , \quad f_j \in H(\Gamma) .$$

Since $\lambda \in \mathrm{Exp}_c(0, \Gamma)$ we know that for every $\varepsilon > 0$, $k \geq 1$ there exists $A > 0$ such that

$$|\lambda(z)| \leq A e^{\varepsilon |z|} \qquad \text{on} \quad \Gamma(k) .$$

Thus, using the interpolation results from [2] we see that for any k there are $\alpha_j \in H(S(\rho; k, \varepsilon_k, \alpha_k))$ such that

$$\lambda = \sum_{j=1}^{n} \alpha_j \rho_j ,$$

and

$$|\alpha_j(z)| \leq C_k e^{D_k |z|}$$

for some constants C_k, $D_k > 0$. Replacing now Theorem 2.6 from [7] with the analogous one which follows from Theorem 3.2 above, we can, as in [2; Theorem 4.2], substitute the α_j by β_j globally defined and which satisfy

$$|\beta_j(z)| \leq E_k e^{F_k|z|} \quad \text{in} \quad \Gamma(k).$$

This concludes the proof. □

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