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Geometria algebrica. — *Stability of pencils of plane quartic curves.* Nota di EDOARDO BALLICO (*) e PAOLO OLIVERIO (*), presentata (**) dal Corrisp. E. VESENTINI.

RIASSUNTO. — In questa nota si danno dei criteri per la stabilità di fasci di quartiche piane.

INTRODUCTION

In this note we study the classification of pencils of quartic curves in $\mathbf{P}^2$ (over an algebraically closed field k of characteristic 0), up to projective equivalence. The general problem of the classification of all pencils of quartic curves in $\mathbf{P}^2$ (or of cubic curves as well) is not covered here: it seems to be very long, boring and probably useless. Our aim is more restricted: simply we give a criterion foı a pencil of quartic curves in $\mathbf{P}^2$ to be unstable or semistable or properly stable in the sense of Mumford's Geometric Invariant Theory [2]. Thus we extend the work of R. Miranda [1] on pencils of cubics. The criterions are actually sufficiently powerful and expressed in a geometric language to say in a few minutes if a given pencil is unstable or properly stable. In particular we have given many examples of properly stable pencils with a fixed component, even a cubic as component. Perhaps this work may be applied to invariant theory of pencils of curves of genus 3, since every non-hyperelliptic curve of genus 3 has a canonical embedding in $\mathbf{P}^2$ as a quartic curve. But this part is not so easy, since there are a lot of properly stable pencils of quartic curves without non singular members. Thus this problem is not considered here.

0. PRELIMINAIRES

Let k be an algebraically closed field with $ch(k) = 0$. Let V be a 3-dimensional vector space over k and x, y, z a basis of V. The projective space $\mathbf{P}^{14} \cong \mathbf{P}(\mathrm{S}^4 \mathrm{V}^*)$ is the parameter space for the quartic curves in $\mathbf{P}^2$. Let $\mathrm{G}(1, 14)$ be the Grassmann variety of line in $\mathbf{P}^{14}$; a point of $\mathrm{G}(1, 14)$ corresponds to a pencil of quartic curves in $\mathbf{P}^2$. $\mathrm{G}(1, 14)$ is naturally embedded in $\mathbf{P}^{104} \cong \mathbf{P}(\Lambda^2 \mathrm{S}^4 \mathrm{V}^*)$ via the Plucher co-ordinates. Let $\mathrm{A} = \Sigma a_{ij} x^i y^j z^{4-i-j}$ and $\mathrm{B} = \Sigma b_{ij} x^i y^j z^{4-i-j}$; the pencil spanned by A and B has Plucher coordinates $m_{ijkl} = a_{ij} b_{kl} b_{ij}$. The algebraic group $\mathrm{SL}(3) = \mathrm{SL}(\mathrm{V})$ acts

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linearly on V, V^*, $G(1, 14)$ and $\mathbf{P}^{104}$. If $g \in SL(3)$ is in a diagonal form with $gx = ux$, $gy = vy$, $gz = wz$, we have $g(m_{ijkl}) = u^{i+k} v^{j+l} w^{8-i-j-k-l} m_{ijkl}$. Now we need the standard definitions and results of Mumford's Geometric Invariant Theory. For more details and the proofs, see [1], [2], [3], [4]. Let G be a reductive algebraic group defined over k, $ch(k) = 0$ (for example $SL(r)$, $r \geq 1$). Let W be an n-dimensional representation of G, and let x be a vector in W. Let $G \cdot x$ denote the orbit and G_x the stabilizer of x.

Definition 1.

i) x is unstable of $0 \in \overline{G \cdot x}$.

ii) x is semi-stable if $0 \notin \overline{G \cdot x}$ i.e. if x is not unstable.

iii) x is properly stable if $G \cdot x$ is closed and G_x is finite.

iv) x is strictly semi-stable if x is semi-stable but not properly stable.

Let $\mathbf{P}(W^*)$ be the projective space of 1-dimensional subspaces of W. A point p of $\mathbf{P}(W^*)$ is called unstable (semi-stable or properly-stable) if any non-zero vector x of W representing p is unstable (semi-stable or properly-stable respectively).

Let W_S and W_{SS} be the open cones of properly stable and semistable vectors of W, and let $\mathbf{P}_S(W^*)$, $p_{SS}(W^*)$ be the open sets of stable and semi-stable points of $\mathbf{P}(W^*)$.

Theorem [2, Theorem 1.10]. *Let* W *be an n-dimensional representation of* G, *inducing an action of* G *on* $\mathbf{P}(W^*)$. *Let* $Y \subset \mathbf{P}(W^*)$ *be a closed* G-*invariant subscheme of* $\mathbf{P}(W^*)$; Y *is then a projective scheme on which* G *acts. Put* $Y_{SS} = Y \cap \mathbf{P}_{SS}(W^*)$ *and* $Y_S = Y \cap \mathbf{P}_S(W^*)$. *Then*:

a) *an universal categorial quotient* (X, π) *of* Y_{SS} *by* G *exists and* X *is a projective scheme*;

b) *there is an open set* X_S *of* X *such that* $(X_S, \pi \mid Y_S)$ *is a univesal geometric quotient of* Y_S *by* G.

In particular the points of $X_S = \pi(Y_S)$ classify the orbits in Y_S; two orbits in Y_{SS} are identified in X if and only if they have the same closure in Y_{SS}.

Stability is not only an interesting notion for invariant theory, it is also a computable one in many interesting cases. This is due to the existence of a strong numerical criterion for stability due Mumford (and in particular cases to Hilbert).

Definition 2. Let W be a linear representation of $GL(1)$. Since $GL(1)$ is reductive W splits into a direct sum of eigenspaces $W = \bigoplus_{n \in \mathbf{Z}} W_n$ where the action of $GL(1)$ on W_n is given by the scalar multiplication by t^n. For any $x \in W$ we have $x = \Sigma x_n$ with $x_n \in W_n$. The weights of x with respect to this representation are the set of integers n such that $x_n \neq 0$.

DEFINITION 3. Let W be a representation of an algebraic group G. Let $\lambda : \mathrm{GL}(1) \rightarrow \mathrm{G}$ be a 1-parameter subgroup of G; and let x be a point of W. The λ-weights of x are the weights of x with respect to the representation of GL (1) on W induced by λ. The same terminology is used for a point p in $\mathbf{P}(\mathrm{W}^*)$.

Now we state the fundamental numerical criterion of stability.

THEOREM 2. [2, THEOREM 2.1]. *Let* G *be a reductive algebraic group acting linearly on the vector space* V *and let* x *be a vector in* V. *Then*:

a) x *is unstable if and only if there exists a* 1*-parameter subgroup* λ *of* G *such that the* λ*-weights of* x *are all positive*;

b) x *is semistable if and only if no such* 1*-parameter subgroups of* G *exists*;

c) x *is properly stable if and only if* x *has both positive and negative weights for every non-trivial* 1*-parameter subgroup of* G;

d) $\mathrm{G}\cdot x$ *is closed* (*i.e. in the terminology of* [2] x *is stable*) *if and only if for every* 1*-parameter subgroup* λ *of* G, *either the* λ*-weights of* x *are both positive and negative or* 0 *is the only* λ*-weight of* x *i.e.* λ *stabilizes* x.

Theorem 2 means in particular that an orbit $\mathrm{G}\cdot x$ *is closed if and only if for every* 1*-parameter subgroup* λ *of* G *the* λ*-orbit of* x *is closed. Since every action of* GL (1) *on a vector space can be diagonalized, Theorem 2 is an effective tool as we shall see in the next section 1. Now we are able to say when a pencil* H *of quartic curves is unstable or strictly semistable. By Theorem 2 a pencil* H *is unstable if and only if there exists a* 1*-parameter subgroup* $\lambda : \mathrm{GL}(1) \rightarrow \mathrm{SL}(3)$ *such that the weights of* H *with respects to* λ *are all positive. Suppose this happens.*

We may choose a basis x, y, z of V such that in this basis λ has a diagonal form: $\lambda(t)(x, y, z) = (t^{r_x} x, t^{r_y} y, t^{r_z} z)$ with after permuting the coordinates, $r_x \geq _y \geq r_z$, $r_x \geq 0$ and $r_x + r_y + r_z = 0$. We have $\lambda(t)(m_{ijkl}) = (t^{r_x(i+k)+r_y(j+l)+r_z(8-i-j-k-l)} m_{ijkl})$. The weights of the points $(m_{ijkl}) \in \mathbf{P}^{104}$ with respect to λ are the exponents of t in (1) for which m_{ijkl} is non zero. By setting $r_z = -r_x - r_y$ the exponent is:

$$r_x(2i + 2k + j + l - 8) + r_y(2j + 2l + i + k - 8).$$

We have $r_x \geq r_y \geq -r_x - r_y$; put $r = r_y/x_x$; r is a rational number with $-1/2 \leq r \leq 1$.

Consider the map $e_{ijkl} : [-1/2, 1] \cap \mathbf{Q} \rightarrow \mathbf{Q}$ given by $e_{ijkl}(r) = (2i + 2k + j + l - 8) + r(2j + 2l + i + k - 8)$. Now from Theorem 2 we immediately obtain the following criterion:

PROPOSITION 1. H *is an unstable* (*resp. non properly stable*) *pencil if and only if there exists a rational number* $r \in [-1/2, 1]$ *and coordinates in* $\mathbf{P}^2$ *such that if* H *is represented by the point* (m_{ijkl}) *in these coordinates* $m_{ijkl} = 0$ *whenever* $e_{ijkl}(r) \leq 0$ (*resp.* $e_{ijkl}(r) < 0$). *The conditions* $e_{ijkl}(r) \leq 0$ *or* $e_{ijkl}(r) < 0$

for all i, j, k, l *subdivide the interval* $[-1/2, 1]$ *into then intervals with end points* $-1/2$, $-2/5$, $-1/3$, $-1/4$, $-1/7$, 0, $1/6$, $-1/3$, $1/2$, $2/3$, 1. *Furthermore not all conditions are independent. It is sufficient to check the condition of Proposition* 1 *for instability for only one rational number* r *in each of the intervals* $(-1/2, -2/5)$, $(-1/7, 0)$, $(1/3, 1/2)$, $(1/2, 2/3)$; *for non proper stability we have to consider also* $r = 0$ *and* $r = 1/2$. *Thus we obtain the following explicit criterion.*

PROPOSITION 2. *A pencil* H *of quartic curves is unstable if and only if there exist homogeneous coordinates in* $\mathbf{P}^2$ *such that if* (m_{ijkl}) *are induced Plucher coordinates of* H, *then one of the following sets vanishes*:

Case a) m_{0001}, m_{0002}, m_{0003}, m_{0004}, m_{0010}, m_{0011}, m_{0012}, m_{0013}, m_{0020}, m_{0021}, m_{0022}, m_{0102}, m_{0103}, m_{0104}, m_{0110}, m_{0111}, m_{0112}, m_{0113}, m_{0120}, m_{0121}, m_{0122}, m_{0203}, m_{0204}, m_{0210}, m_{0211}, m_{0212}, m_{0213}, m_{0220}, m_{0221}, m_{0222}, m_{0304}, m_{0310}, m_{0311}, m_{0312}, m_{0313}, m_{0320}, m_{0321}, m_{0322}, m_{0410}, m_{0411}, m_{0412}, m_{0413}, m_{0420}, m_{0421}, m_{0422}, m_{1011}, m_{1012}, m_{1013}, m_{1112}, m_{1113}, m_{1213}.

Case b) m_{0001}, m_{0002}, m_{0003}, m_{0004}, m_{0010}, m_{0011}, m_{0012}, m_{0013}, m_{0020}, m_{0021}, m_{0022}, m_{0030}, m_{0031}, m_{0102}, m_{0103}, m_{0104}, m_{0110}, m_{0111}, m_{0112}, m_{0113}, m_{0120}, m_{0121}, m_{0022}, m_{0030}, m_{0031}, m_{0102}, m_{0103}, m_{0104}, m_{0110}, m_{0111}, m_{0112}, m_{0113}, m_{0120} m_{0121}, m_{01_2}, m_{0130}, m_{0203}, m_{0204}, m_{0210}, m_{0211}, m_{0212}, m_{0213}, m_{0220}, m_{0221}, m_{0222}, m_{0304}, m_{0310}, m_{0311}, m_{0312}, m_{0313}, m_{0320}, m_{0321}, m_{0401}, m_{0411}, m_{0412}, m_{0420}, m_{1011}, m_{1012}, m_{1013}, m_{1020}, m_{1020}, m_{1021}, m_{1121}, m_{1113}, m_{1120}.

Case c) m_{0001}, m_{0002}, m_{0003}, m_{0004}, m_{0010}, m_{0011}, m_{0012}, m_{0013}, m_{0020}, m_{0021}, m_{0022}, m_{0030}, m_{0031}, m_{0040}, m_{0102}, m_{0103}, m_{0104}, m_{0110}, m_{0111}, m_{0112}, m_{0113}, m_{0120}, m_{0121}, m_{0122}, m_{0130}, m_{0131}, m_{0203}, m_{0204}, m_{0210}, m_{0211}, m_{0212}, m_{0220}, m_{0221}, m_{0230}, m_{0310}, m_{0311}, m_{0320}, m_{0410}, m_{1011}, m_{1012}, m_{1013}, m_{1020}, m_{1021}, m_{1022}, m_{1030}, m_{1112}, m_{1120}, m_{1121}, m_{1220}.

Case d) m_{0001}, m_{0002}, m_{0003}, m_{0004}, m_{0010}, m_{0011}, m_{0012}, m_{0013}, m_{0020}, m_{0021}, m_{0022}, m_{0030}, m_{0031}, m_{0040}, m_{0102}, m_{0103}, m_{0104}, m_{0110}, m_{0111}, m_{0112}, m_{0113}, m_{0120}, m_{0121}, m_{0122}, m_{0130}, m_{0131}, m_{0140}, m_{0203}, m_{0210}, m_{0211}, m_{0212}, m_{0220}, m_{0221}, m_{0230}, m_{0310}, m_{0311}, m_{0320}, m_{0410}, m_{1011}, m_{1012}, m_{1013}, m_{1020}, m_{1021}, m_{1022}, m_{1030}, m_{1031}, m_{1112}, m_{1120}, m_{1121}, m_{1130}, m_{1220}, m_{2021}.

In case a) any $r \in (-1/2, -2/5)$ will show that H is unstable, in case *b*) we take $r \in (-1/7, 0)$, in case *c*) $r \in (-1/3, 1/2)$ and in case *d*) $r \in (1/2, 2/3)$.

PROPOSITION 3. *A pencil* H *is not properly stable if and only if either it is unstable or there exist homogeneous coordinates of* $\mathbf{P}^2$ *such that if* (m_{ijkl}) *are the Plücher coordinates of* p, *then one of the following sets of* m_{ijkl}*'s vanishes*:

Case e) $m_{0001}, m_{0002}, m_{0003}, m_{0004}, m_{0010}, m_{0011}, m_{0012}, m_{0013}, m_{0020}, m_{0021}, m_{0022}, m_{0030}, m_{0031}, m_{0102}, m_{0103}, m_{0104}, m_{0110}, m_{0111}, m_{0112}, m_{0113}, m_{0120}, m_{0121}, m_{0122}, m_{0130}, m_{0203}, m_{0204}$ m_{0210} $m_{0211}, m_{0212}, m_{0213}, m_{0220}, m_{0221}, m_{0304}, m_{0310}, m_{0311}, m_{0312}, m_{0320}, m_{0410}, m_{0411}, m_{1011}, m_{1012}, m_{1013}, m_{1020}, m_{1021}, m_{1112}, m_{1120}$.

Case f) $m_{0001}, m_{0002}, m_{0003}, m_{0004}, m_{0010}, m_{0011}, m_{0012}, m_{0013}, m_{0020}$ $m_{0021}, m_{0022}, m_{0030}, m_{0031}, m_{0040}, m_{0102}, m_{0103}, m_{0104}, m_{0110}, m_{0111}, m_{0112}, m_{0113}, m_{0120}, m_{0121}, m_{0122}, m_{0130}, m_{0131}, m_{0203}, m_{0210}, m_{0211}, m_{0212}, m_{0220}, m_{0221}, m_{0230}, m_{0310}, m_{0311}, m_{0320}, m_{0410}, m_{1011}, m_{1012}, m_{1013}, m_{1020}, m_{1021}, m_{1022}$ m_{1030} m_{1112} $m_{1120}, m_{1121}, m_{1220}$.

For case *e*) we consider $r = 0$, for case *f*) we consider $r = 1/2$.

Now we can give a more geometric form of the conditions above for instability and non properly stability. In fact we show that we may choose coordinates and 2 generators A, B for the pencil H such that the equations f_A, f_B of A and B are very simple and in particular they must have a lot of zero coefficients. We have 15 monomials of degree 4 in the variables x, y, z; a notation $g \in (w_1, \cdots, w_S)$ where w_i is a monomial of degree 4 in the variables x, y, z means that q is in the linear span of the monomials $w_1, \cdots, w_S$; a notation $g \in (w_1, \cdots, w_S)$ where w_i is a monomial of degree 4 in x, y, z means that g is in the linear span of the monomials of degree 4 different from $w_1, \cdots, w_S$.

Take two quartics A, B generating an unstable (or non properly stable) pencil H. Choose coordinates x, y, z such that one of the vanishing conditions of Proposition 2 (or of proposition 3) is satisfied. Consider the equation $f_A = \Sigma a_{ij} x^i y^j z^{4-i-j}$, $f_B = \Sigma b_{ij} x^i y^j z^{4-i-j}$ of A and B. Since $m_{ijkl} = a_{ij} b_{kl} - a_{kl} b_{ij}$, each set of vanishing conditions gives equations involving the coefficient a_{ij} and b_{kl}.

These equations are easily seen to be equivalent to the vanishing of certain of the coefficients of some pair of quartics A′, B′ in the pencil H. The proof consists in high school algebraic manipulations and thus it is omitted. As an example of how to do the calculation we consider the condition given by the first 13 coefficient in case b): m_{00kl} with $(k, l) \neq (4, 0)$. Either $a_{00} = b_{00} = 0$ (and thus this happens for every quartic in the pencil) or one of them say a_{00} is $\neq 0$; considering $B' = B - (b_{00}/a_{00}) A$, we may suppose $b_{00} = 0$; the conditions m_{00kl}'s shows $b_{kl} = 0$ if $(k, l) \neq (4, 0)$ i.e. $B \in (x^4)$; in the first case we consider the other coefficient m_{01kl} in case b) and so on.

PROPOSITION 4. *A pencil* H *of quartic plane curves is unstable if and only if there exist coordinates* x, y, z *in* $\mathbf{P}^2$ *and generators* $A, B \in H$ *with equations*

f_A, g_B *such that one of the following conditions is satisfied:*

1) $f_B \in (x^4, x^3 y)$, *no restriction for* f_A;

2) $f_B \in (x^2 z^2, x^2 yz, x^2 y^2, x^3 z, x^3 y, x^4)$, $f_A \in)z^4, yz^3, y^2 z^2, y^3 z, y^4($;

3) $f_B \in (x^2 yz, x^2 y^2, x^3 z, x^3 y, x^4)$, $f_A \in)z^4 yz^3, y^2 z^2, y^3 z, xz^3($;

4) $f_B \in (x^2 z^2, xy^3, x^2 yz, x^2 y^2, x^3 z, x^3 y, x^4)$, $f_A \in)xy^2 z, xy^3, x^2 z^2, x^2 yz,$ $x^2 y^2, x^3 z, x^3 y, x^4($;

5) $f_B \in (y^4, xy^3, x^2 y^2, x^3 z, x^3 y, x^4)$, $f_A \in)z^4, yz^3, y^2 z^2, xz^3($;

6) $f_B \in (y^4, xy^3, x^2 yz, x^2 y^2, x^3 z, x^3 y, x^4)$, $f_A \in)z^4, yz^3, y^2 z^2, xz^3, xyz^2($;

7) $f_A, f_B \in (y^3 z, y^4, xy^2 z, xy^3, x^2 yz, x^2 y^2, x^3 z, x^3 y, x^4)$;

8) $f_B \in (y^4, xy^3, x^2 y^2, x^3 y, x^4)$; $f_A \in)z^4, yz^3, xz^3($.

1) and 2) come from case a). 3) and 4) from case b), 5), 6) and 7) from case c) and 8) from case d).

Furthermore particular cases of 1) satisfy conditions of case b) or case c) or case d). A particular case of 4) satisfies the conditions of case a). A particular case of 7) satisfies the conditions of d) and a particular case of 8) comes from d).

PROPOSITION 5. *A pencil* H *of quartic plane curves is properly stable if and only if either it is unstable or there exist homogeneous coordinates* x, y, z *in* $\mathbf{P}^2$ *and generators* $A, B \in H$ *with equations* f_A, f_B *such that one of the following conditions is satisfied:*

9) $f_B \in (x^2 y^2, x^3 z, x^3 y, x^4)$, $f_A \in)z^4, yz^3($;

10) $f_B \in (xy^3, x^2 yz, x^2 y^2, x^3 y, x^3 z, x^4)$, $f_A \in)z^4, yz^3, y^2 z^2, xz^3($;

11) $f_B \in (y^4, xy^2 z, xy^3, x^2 yz, x^2 y^2, x^3 z, x^3 y, x^4)$, $f_A \in)y^4, xy^2 z, xy^3,$ $x^2 z^2, x^2 yz, x^2 y^2, x^3 z, x^3 y, x^4($;

12) $f_B \in (y^4, x^2 yz, x^2 y^2, x^3 z, x^3 y, x^4)$, $f_A \in)z^4, yz^3, y^2 z^2, xz^3, xyz^2($.

When one translates Proposition 3 into Proposition 5, one realizes that many pencils which satisfy e) or f) satisfy also a) or b) or c) and thus they are unstable; for this reason we have only 4 cases of semistable but not properly stable pencils.

2. Proposition 4 and 5 give effective rules for checking if a given pencil is unstable or properly stable. We will apply them to classification of pencils with closed orbits but infinite stabilizer (in particular pencils with closed orbits but infinite stabilizer) and to give examples of pencils properly stable with fixed component or without smooth elements. First we need a "coordinate-free" translation of Proposition 4 and 5. In each case we consider only the "general

pencil ". The other pencils have at the base points singularities as least as bad as in the general case and eventually higher order of contact for 2 quartics in the pencils:

1) B is irreducible and has a triple line as a component; there is no restriction for A.

2) B is the union of a conic and a double line L ; A is the union of L and a cubic.

3) B has a double line L as a component; L has a singular point $P \in L$ and L has contact of order 4 with A at the point P.

4) B and A have a line L as component; L has contact of order 3 with L.

5) B has a triple point P with a triple line L as tangent cone; A has a singular point at P with L in the tangent cone of A at P.

6) B has a triple point P and the tangent cone to B at P contains a double line L; A has a cusp a P with L as tangent cone.

7) A and B have a common triple point.

8) B is the union of 4 lines through a point P and A has a double point at P.

9) B is the union of a double line L and a conic tangent to L at the point P; A is tangent to L at P.

10) $B = L \cup E$ with L a line containing a point $P \in E$ and E singular at P with L in the tangent cone; A has a double point at P with L in the tangent cone.

11) A has a cusp at a point P and the line L in the tangent cone to A at P has contact of order at least 4 with A; B has a triple point at P and L has contact of order ≥ 4 at P.

12) B has a triple point at P with a double line L in the tangent cone, has a cusp at P with L as the tangent cone.

THEOREM 3. *In a semistable pencil of quartic plane curves the generic element has at most double points*; *there exist properly stable pencils which contain only singular curves and even only reducible curves.*

The proof of theorem follows from the description above. In fact by Bertini's theorem the general element of a pencil is singular only at the base points. If at a point P every element of a pencil has a triple point, then L is not properly stable because it satisfies the conditions of 7) and thus the pencil is unstable. For the last part of the theorem it is easy to show when a pencil with a base curve is unstable or properly stable. We consider here only the general case. Let H be a pencil quartic curve with a base cubic curve C and a base point $0 \notin C$. Then H is formed by the curves $C \cup L$ with L line though 0. Suppose that C has only ordinary double points (eventually C may be reducible). Then

H is not unstable because it is easy to see that it does not satisfy any condition 1), ⋯, 8) or their specializations. For examples, in the case 8) no curve in H has a 4-ple point. Now suppose that H is a pencil with a reduced conic C and 4 points P_1, P_2, P_3, P_4 as base locus. Then 3 of the P_i's are collinear. If C is non singular then H is semistable. If C is reducible then H is unstable if and only if there are 2 lines L_1, L_2 containing the singular point of C and the points P_1, P_2, P_3, P_4; if this is the case, the pencil satisfies the condition 8). Now suppose that a pencil H has a base line L and 9 base points $P_1, \cdots, P_9$ with no 3 P_i's collinear and with $P_1 \cup \cdots \cup P_9$ complete intersection of 2 smooth cudics. Then the pencil is properly stable. If the base locus of a pencil H is a smooth cubic C and a point $0 \notin C$ such that no flex of C passes through 0, H is properly stable. If the base locus is a smooth conic and 4 points not on it, then the pencil is properly stable. If the pencil H has 3 lines not through the same point and a point 0 not on the 3 lines, then H is a semistable as we have just said, and has finite stabilizers but it has not a close orbit; in fact every pencil with 3 lines in the base locus is in the closure of the orbit.

Now we want to see when a pencil has no finite stabilizer and when it has a closed orbit.

PROPOSITION 6. *The pencil* H *has a closed orbit and finite stabilizer if and only if there exist homogenous coordinates such that* H *has all Plucher coordinates zero except eventually*:

i) $m_{0222}, m_{0230}, m_{1022}, m_{1030}$;

ii) $m_{1121}, m_{1113}, m_{0321}, m_{0313}$;

iii) $m_{1220}, m_{0412}, m_{0420}$;

with at last two m_{ijkl}'s $\neq 0$ and $m_{0222} \neq 0$ (resp. $m_{1112} \neq 0$, $m_{1220} \neq 0$) in case *i*) (resp. *ii*), (iii)). In the case i) we have $f_A = uy^3 z + vxz^3$, $f_B = wx^2 y^2 + mx^3 z$ with $uw \neq 0$ and v or m not zero. In the case *ii*) we have $f_A = uy^2 z^2 + vxz^3$, $f_B = wx^2 yz + mxy^3$ with $uw \neq 0$ and v or m not zero. In the case iii) we have $f_A = uxy^2 z + vy^4$, $f_B = wx^2 z^2 + my^4$ with u, v, w, m as above. If H is a pencil stabilized by a 1-parameter subgroup and not as above, then there exist homogeneous coordinates such that H has only 1 not zero m_{ijkl}; thus H is unstable.

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