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The solution operator for a partial differential equation with delay

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Analisi matematica. — *The solution operator for a partial differential equation with delay* (*). Nota di GABRIELLA DI BLASIO (**), KARL KUNISCH (***) e EUGENIO SINESTRARI (****), presentata (*****) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Viene dimostrata l'esistenza e l'unicità globale della soluzione di un'equazione funzionale in uno spazio di Hilbert e si caratterizza il generatore infinitesimale del semigruppo ad essa associato. Il risultato è applicato ad equazioni integrodifferenziali a derivate parziali di tipo parabolico in cui compaiono argomenti con ritardo (discreto e continuo) nelle derivate spaziali di ordine massimo.

1. INTRODUCTION

In this paper we shall study a class of partial differential equations with deviating argument in the time variable. As an example of this class we can consider the following

$$
(1) \qquad \begin{cases} u_t(t,x) = u_{xx}(t,x) + u_{xx}(t-r,x) + \displaystyle\int_{-r}^{0} a(s)\, u_{xx}(t+s,x)\, ds, \\ t \geq 0, \qquad 0 \leq x \leq 1 \\ u(t,0) = u(t,1) = 0, \qquad t \geq 0 \end{cases}
$$

where $r > 0$ is given. In this paper we shall study problem (1) in a space of L^2 functions with respect to t. Therefore it is known that we must impose a pair of initial data

$$
(2) \qquad u(t,x) = \eta(t,x) \quad \text{a.e.} \quad t \in]-r,0[\quad , \quad u(0,x) = \xi(x).
$$

We shall prove that if η is square integrable from $]-r,0[$ into $H^{2,2}(0,1) \cap H_0^{1,2}(0,1)$ and $\xi \in H_0^{1,2}(0,1)$, then there exists a unique global solution of (1), (2). Moreover we shall prove that these solutions generate a C_0-semigroup in the product space $H_0^{1,2}(0,1)\, x L^2(-r,0\,;H^{2,2}(0,1) \cap \cap H_0^{1,2}(0,1))$. The characterization of its infinitesimal generator is also given. This choice of state space was introduced by Di Blasio in [4]

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where (1) is considered with $a = 0$. Partial differential equations with delay in the highest order derivatives have been studied in different state spaces by Travis and Webb [9], Ardito and Ricciardi [1] and Kunisch and Schappacher [6].

2. EXISTENCE AND UNIQUENESS RESULTS

To study problem (1), (2) we shall rewrite it as an abstract functional differential equation. To this end we will introduce a Hilbert space H with norm $\|\cdot\|$ and the linear operators A, L_1 and L_2 satisfying the following properties

(H_1) $A : D_A \subset H \to H$ generates an analytic and bounded semigroup $\exp(tA)$ on H

(H_2) L_1 is continuous from D_A (endowed with the graph norm $\|x\|_{D_A} = \|x\| + \|Ax\|$) into H

(H_3) L_2 is continuous from $L^2(-r, 0; D_A) = \mathcal{H}$ into H.

Moreover we shall denote by F the following interpolation space between D_A and H (see Lions [7])

$$F \{ x = u(0) : u \in L^2(0, \infty; D_A) \cap W^{1,2}(0, +\infty; H) \}$$

endowed with the norm (see Butzer and Berens [2])

$$\|x\|_F = \|x\| + \left(\int_0^{+\infty} \|A \exp(tA)\, x\|^2 \, dt \right)^{\frac{1}{2}} .$$

Moreover we recall (see Lions and Magenes [8]) that if $u \in L^2(0, T; D_A) \cap W^{1,2}(0, T; H)$ then $u \in C(0, T; F)$ and we have

$$\|u(t)\|_F \le c_1 \|u\|_{L^2(0,T;D_A) \cap W^{1,2}(0,T;H)} \, . \tag{3}$$

Now let us denote by Z the Hilbert space $Z = F \times \mathcal{H}$; given $(x, y) \in Z$ we shall consider the following problem

$$\begin{cases} u'(t) = Au(t) + L_1 u(t-r) + L_2 u_t , & t \ge 0 \\ (u(0), u_0) = (x, y) \end{cases} \tag{4}$$

where for each $t \ge 0$ $u_t : [-r, 0] \to H$ is defined as $u_t(\theta) = u(t + \theta)$.

We have

THEOREM 1. *For each* $(x, y) \in Z$ *and* $T > 0$ *there exists a unique* $u \in L^2(-r, T; D_A) \cap W^{1,2}(0, T; H)$ *satisfying (4). Moreover there exists* c_2 *(depending on* T*) such that*

$$\|u\|_{L^2(0,T;D_A) \cap W^{1,2}(0,T;H)} \le c_2 (\|x\|_F + \|y\|_{L^2(-r,0;D_A)}) \, . \tag{5}$$

Furthermore $u \in C(0, T; F)$ and we have

$$\| u(t) \|_F \leq c_1 c_2 (\| x \|_F + \| y \|_{L^2(-r,0;D_A)}) . \tag{6}$$

Proof. Let $T < r$ and consider the integrated version of (4)

$$u(t) = \exp(tA)\, x + \int_0^t \exp((t-s)A)(L_1 y(s-r) + L_2 \tilde{u}_s)\, ds = (Qu)(t) \tag{7}$$

where we have set

$$\tilde{u}(t) = \begin{cases} y(t) & , \quad -r < t < 0 \\ u(t) & , \quad t \geq 0 . \end{cases}$$

Now using [8], vol. II, Thm. 3.2 it can be proved that Q maps $L^2(0, T; D_A)$ into itself and moreover we have

$$\| Qu - Qv \|_{L^2(0,T;D_A)} \leq \text{const} \left(\int_0^T \| L_2 (\tilde{u}_s - \tilde{v}_s) \|^2 ds \right)^{\frac{1}{2}}$$

and hence using assumption (H_3)

$$\| Qu - Qv \|_{L^2(0,T;D_A)} \leq \text{const} \left(\int_0^T \| L_2 \| \left(\int_{-r}^0 \| \tilde{u}(s+\theta) - \tilde{v}(s+\theta) \|_{D_A}^2 d\theta \right)^2 ds \right)^{\frac{1}{2}} =$$

$$\text{const} \left(\int_0^T \int_0^s \| u(\theta) - v(\theta) \|_{D_A}^2 d\theta\, ds \right)^{\frac{1}{2}} \leq$$

$$\text{const}\, T^{\frac{1}{2}} \| u - v \|_{L^2(0,T;D_A)} .$$

Therefore if T is sufficiently small, Q is a strict contraction in $L^2(0, T; D_A)$. Consequently there exists a unique $u \in L^2(0, T; D_A)$ which satisfies (7) and hence, using [8], vol. II, Thm. 3.2 once again, we find that there exists a unique $u \in L^2(0, T; D_A) \cap W^{1,2}(0, T; H)$ satisfying (4) and (5). Furthermore estimate (6) is a consequence of (3). Finally the result for all T can be proved by iteration.

The following theorem establishes further properties of the solutions of (4).

THEOREM 2. *Let* $(x, y) \in Z$ *be such that*

$$y \in W^{1,2}(-r, 0; D_A),\; y(0) = x,\; Ax + L_1 y(-r) + L_2 y \in F . \tag{8}$$

Then the solution of problem (4) belongs to $W^{1,2}(-r, T; D_A) \cap \cap W^{2,2}(0, T; H)$, for each $T > 0$.

Proof. Let Q be the operator introduced in the proof of Theorem 1. It can be seen that if (x, y) satisfies (8) then Q is a strict contraction in $W^{1,2}(0, T; D_A)$, if T is sufficiently small. Consequently there exists a unique $u \in W^{1,2}(0, T; D_A)$ satisfying (7) and hence, using Phillips' theorem (see Kato [5]), it can be proved that there exists a unique $u \in W^{1,2}(-r, T; D_A) \cap W^{2,2}(0, T; H)$ satisfying (4). The result for all T can be proved by iteration.

3. THE INFINITESIMAL GENERATOR OF THE SOLUTION OPERATOR

For each $z = (x, y) \in Z$ let us denote by u the solution of (4). Moreover let us define for each $t \geq 0$ the following linear operator on Z

$$S(t) z = (u(t), u_t).$$

We have

THEOREM 3. *S(t) is a C_0 semigroup on Z, i.e. satisfies the following properties*

(i) $S(t) \in \mathscr{L}(Z, Z)$, *for each* $t \geq 0$

(ii) $S(0) = I$ (*identity operator*)

(iii) $S(t+s) z = S(t) S(s) z$, *for each* $z \in Z$ *and* $t, s \geq 0$

(iv) $\lim_{t \to 0} S(t) z = z$, *for each* $z \in Z$.

Proof. Assertion (i) is a consequence of (5). Moreover (ii) is evident and (iii) follows from the uniqueness of the solutions of (4). Furthermore we have

$$\| S(t) z - z \|_Z^2 = \| u(t) - x \|_F^2 + \int_{-r}^{0} \| u(t+\theta) - y(\theta) \|_{D_A}^2 \, d\theta .$$

Therefore (iv) follows from the fact that $u \in C(0, T; F) \cap L^2(-r, T; D_A)$, for each $T > 0$.

Now let us denote by $\Lambda : D_\Lambda \subset Z \to Z$ the operator defined as follows

$$D_\Lambda = \{(x, y) \in Z : y \in W^{1,2}(-r, 0; D_A), y(0) = x, Ax + L_1 y(-r) + L_2 y \in F\}$$

$$\Lambda(x, y) = (Ax + L_1 y(-r) + L_2 y, y').$$

We shall prove that Λ is the infinitesimal generator of the semigroup S. To this end we prove the theorem:

THEOREM 4. *The following properties hold*:

(i) $S(t) D_\Lambda \subset D_\Lambda$, *for each* $t \geq 0$

(ii) $\lim_{t \to 0} \dfrac{S(t) z - z}{t} = \Lambda z$, *for each* $z \in D_\Lambda$

(iii) D_Λ is dense in Z

(iv) Λ is a closed operator

Proof. Assertion (i) and (ii) can be proved by using Theorem 2. Assertion (iii) follows from noting that for each $z \in Z$ we have

$$\int_0^t S(s)\, z \, ds \in D_A$$

and

$$\lim_{t \to 0} \int_0^t S(s)\, z \, ds = z .$$

To prove (iv) let $z_n = (x_n, y_n) \in D_A$ be such that

(9) $$Z - \lim z_n = z = (x, y)$$

and that

(10) $$Z - \lim \Lambda z_n = w = (u, v) .$$

We have

$$y_n \to y \qquad \text{in} \quad L^2(-r, 0; D_A)$$

and

$$y_n' \to u \qquad \text{in} \quad L^2(-r, 0; D_A)$$

so that $u = y'$, $y' \in L^2(-r, 0; D_A)$ and moreover

(11) $$y_n \to y \qquad \text{in} \quad W^{1,2}(-r, 0; D_A)$$

from (9) and (11) we get

(12) $$x_n = y_n(0) \to y(0) = x \qquad \text{in} \quad D_A$$

so that from (12) and (10)

$$H - \lim A x_n + L_1 y_n(-r) + L_2 y_n = A x + L_1 y(-r) + L_2 y$$

$$F - \lim A x_n + L_1 y_n(-r) + L_2 y_n = u \in F$$

and (iv) is proved.

Finally from Theorem 4 and Theorem 1.9 of [3] we get

THEOREM 5. *The operator* Λ *is the infinitesimal generator of the semigroup* $S(t)$.

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