Singular non polynomial perturbations of $-\Delta + |x|^2$


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Analisi matematica. — Singular non polynomial perturbations of $-\Delta + |x|^2$ (ii). Nota di FRANCO NARDINI (**) presentata (***) dal Socio G. CIMMINO.

RIASSUNTO. — Si studia la perturbazione dello spettro dell'operatore $-\Delta + |x|^2$ dovuta all'introduzione di un potenziale singolare non polinomiale e si prova che la serie perturbativa del primo autovalore di tale operatore è sommabile secondo Borel.

In recent years many authors [see 4: ch. XII § 4 for an extensive review] have widely studied the singular perturbation of the spectrum of the Schrödinger operator $H_0 = -\Delta + |x|^2$ in $L^2(\mathbb{R}^n)$ obtained when an additional potential $V(\beta)$ of polynomial type is introduced, $V(\beta)$ depends on a (real or complex) parameter $\beta$ and converges pointwise to zero as $\beta \to 0$. In this situation two main results can be proved: the norm convergence of the eigenprojections of the perturbed operator $H_0 + V(\beta)$ towards those of the unperturbed one $H_0$ as $\beta \to 0$, which in particular implies the continuity of the eigenvalues at $\beta = 0$, and the Borel summability of the (Rayleigh–Schrödinger) perturbation series to the eigenvalues of $H_0 + V(\beta)$. These are precisely the results obtained by Auberson [1] for the one-dimensional operator $H_0 = -d^2/dx^2 + x^2$ and the singular non-polynomial potential $V(\beta)(x) = \beta x^4/(1 + \beta x^2)$ $x \in \mathbb{R}$; in this paper we give an extension of the Auberson’s results to the $n$-dimensional case when $H_0 = -\Delta + |x|^2$ and $V(\beta)(x) = \beta f(x)/(1 + \beta g(x))$: here $f$ and $g$ are homogeneous polynomials in $\mathbb{R}^n$ of degree 4 and 2 respectively which are positive as $x \neq 0$. With regard to the proofs we remark that we have obtained the convergence of the eigenprojections as $\beta \to 0$ exploiting only the strong convergence of the resolvents of $H_0 + V(\beta)$ thanks to the results of Vock and Hunziker [5], while the proof of the Borel summability is obtained by a Watson-like theorem [4: th. XII. 21] and the results of [2]; these proofs can be trivially extended to a more general potential $V(\beta)$ analytic with respect to $\beta$ in a sector $\mathscr{A} = \{\beta \in \mathbb{C} ; |\arg \beta| < \delta\}$ (with $\delta \in (\pi/2, \pi)$), real valued for $\beta \in \mathbb{R}^+$ and satisfying the following conditions:

1) $V(\beta)(x) \to 0 \quad \forall x \in \mathbb{R}^n$

2) $\forall \beta \in \mathscr{A} \quad \exists C(\beta) > 0$ such that

$V(\beta)(x) \leq C(\beta) |x|^2$

$\text{Re} V(\beta)(x) \geq -C(\beta) \quad \forall x \in \mathbb{R}^n \quad \forall \beta \in \mathscr{A}$

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iii) \( \arg V(\beta)(x) \in [0, \theta] \) if \( \arg \beta \in [0, \theta] \) \( \forall x \in \mathbb{R}^n \)

\( \arg V(\beta)(x) \in [-\theta, 0] \) if \( \arg \beta \in [-\theta, 0] \) \( \forall x \in \mathbb{R}^n \)

iv) the following expansion holds for every \( N = 1, 2, 3, \ldots \)

\[
V(\beta)(x) = \sum_{j=0}^{N} a_j(x) \beta^j + \beta^{N+1} R_{N+1}(\beta)
\]

where

\[
| a_j(x) | \leq c | x |^{2j+2} \quad \forall x \in \mathbb{R}^n \quad j = 0, 1, 2, \ldots
\]

and

\[
| R_{N+1}(\beta) | \leq c | x |^{2N+4} \quad \forall x \in \mathbb{R}^n .
\]

Before stating our results we recall some definitions. An operator-valued function \( H(\beta) \) defined in a complex domain \( \Omega \) is said to be a holomorphic family of type \( A \) (in the Hilbert space \( \mathcal{H} \)) [3: ch. VII §2.1] if

i) \( H(\beta) \) is a closed operator in \( \mathcal{H} \) with domain \( D(H(\beta)) = D \) independent of \( \beta \in \Omega \),

ii) \( \beta \rightarrow H(\beta)u \) is a (vector-valued) holomorphic function in \( \Omega \) for every \( u \in D \).

The family \( H(\beta) \) is said to be selfadjoint [3: ch. VII §3.1] if \( \Omega \) is symmetric with respect to the real axis, \( D \) is dense in \( \mathcal{H} \) and \( H(\beta)^* = H(\bar{\beta}) \) for every \( \beta \in \Omega \); as for the spectral properties of these families we refer to [3: ch. VII §1.3, 1.5, 3.2]; here we recall only that if \( \lambda_0 \) is an isolated nondegenerate eigenvalue of \( H(0) \) (suppose that \( 0 \in \Omega \)), then there exists an analytic function \( \lambda(\beta) \) defined for \( |\beta| < \delta \) such that \( \lambda(0) = \lambda_0 \) and \( \lambda(\beta) \) is a nondegenerate eigenvalue of \( H(\beta) \) for every \( \beta \) with \( |\beta| < \delta \) [4: th. XII. 3]. The Taylor series of \( \lambda(\beta) \) is called Rayleigh-Schrödinger series and its coefficients can be computed expanding the right hand side of

\[
(\ast)
\lambda(\beta) = \frac{\langle \Omega(0), H(\beta) P(\beta) \Omega(0) \rangle}{\langle \Omega(0), P(\beta) \Omega(0) \rangle};
\]

here

\[
(\ast\ast)
P(\beta) = -(2\pi i)^{-1} \int_{|z-\lambda_0|=\rho} (H(\beta) - z)^{-1} dz
\]

denotes the eigenprojection onto the eigenspace relative to \( \lambda(\beta) \) and \( \Omega(0) \) the eigenvector of \( H(0) \) relative to \( \lambda_0 \).

We say that an operator-valued function \( H(\beta) \) defined in a (real or complex) domain \( \Omega \) converges strongly in the generalized sense to \( H_0 \) as \( \beta \rightarrow \beta_0 \) \( (\beta_0 \in \Omega) \) [3: ch. VIII §1.1] if there exists \( \beta \in C \) such that the resolvents \( R(\beta, z) = = (H(\beta) - z)^{-1} \) of \( H(\beta) \) converge strongly to the resolvent \( R(\beta, z) = (H_0 - z)^{-1} \)
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of $H_0$ as $\beta \to \beta_0$; in these hypotheses an isolated eigenvalue $\lambda_0$ of $H_0$ with finite multiplicity is said to be stable [3: ch. VIII § 1.4] if

i) there exists $\delta > 0$ such that for every $z$ with $0 < |z - \lambda_0| < \delta$ there exists $R(\beta, z)$ for every $\beta$ close to $\beta_0$ and $R(\beta, z)$ converges strongly to $R_0(z)$ as $\beta \to \beta_0$,

ii) if $P(\beta)$ denotes the projection given by $\langle \lambda \rangle (0 < r < \delta)$ and $P_0$ denotes the eigenprojection for the eigenvalue $\lambda_0$ of $H_0$ then $\dim P(\beta) \leq \dim P_0$ for $\beta$ sufficiently close to $\beta_0$.

In what follows we shall denote by $H(\beta)$ the operator $-\Delta + |x|^2 + \beta f(x)/(1 + \beta g(x))$ with domain $D(H(\beta)) = \{ f \in W^1(\mathbb{R}^n); (1 + |x|^2) f \in L^2(\mathbb{R}^n) \}$; we collect our results on $H(\beta)$ in the following theorem.

**Theorem 1.** The operator family $H(\beta)$ $\beta \in \mathbb{C} \setminus (-\infty, 0]$ is selfadjoint holomorphic of type A; it consists of operators with compact resolvents and discrete spectra and converges strongly in the generalized sense to $H(0)$ as $\beta \to 0$ and $|\arg \beta| < \pi - \delta$ ($0 < \delta < \pi/2$); its domain of boundedness is $D_0 = \mathbb{C} \setminus \sigma(H(0))$ and every eigenvalue of $H(0)$ is stable.

It is well known that the least eigenvalue of $H(0)$ is $n$ and is nondegenerate; expanding the right hand side of (\ast) we obtain the coefficients of the so called perturbation series of $\lambda(0) = n$; this series may result divergent for every $\beta \neq 0$ since the family $H(\beta)$ is not holomorphic for $\beta = 0$. Nevertheless the following theorem provides a connection between the perturbation series and the eigenvalue $\lambda(\beta)$ of $H(\beta)$ which converges to $\lambda(0) = n$ as $\beta \to 0$. Referring the reader to [4: ch. XII § 4] for the definition of the Borel summability, we have

**Theorem 2.** There exists $r > 0$ such that the Rayleigh-Schrödinger perturbation series of the least eigenvalue of $H(0)$ is Borel summable to $\lambda(\beta)$ for every $\beta \in \mathbb{C}$ with $|\beta| < r$ and $|\arg \beta| < \delta$.

**References**