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Analisi matematica. — *Singular non polynomial perturbations of* $-\Delta + |x|^2$ (*). Nota di Franco Nardini (**), presentata (***) dal Socio G. Cimmino.

Riassunto. — Si studia la perturbazione dello spettro dell'operatore $-\Delta + |x|^2$ dovuta all'introduzione di un potenziale singolare non polinomiale e si prova che la serie perturbativa del primo autovalore di tale operatore è sommabile secondo Borel.

In recent years many authors [see 4: ch. XII § 4 for an extensive review] have widely studied the singular perturbation of the spectrum of the Schrödinger operator $H_0 = -\Delta + |x|^2$ in $L^2(R^n)$ obtained when an additional potential $V(\beta)$ of polynomial type is introduced, $V(\beta)$ depends on a (real or complex) parameter β and converges pointwise to zero as $\beta \to 0$. In this situation two main results can be proved: the norm convergence of the eigenprojections of the perturbed operator $H_0 + V(\beta)$ towards those of the unperturbed one H_0 as $\beta \to 0$, which in particular implies the continuity of the eigenvalues at $\beta = 0$, and the Borel summability of the (Rayleigh–Schrödinger) perturbation series to the eigenvalues of $H_0 + V(\beta)$. These are precisely the results obtained by Auberson [1] for the one-dimensional operator $H_0 = -d^2/dx^2 + x^2$ and the singular non-polynomial potential $V(\beta)(x) = \beta x^4/(1 + \beta x^2)$ $x \in R$; in this paper we give an extension of the Auberson's results to the n-dimensional case when $H_0 = -\Delta + |x|^2$ and $V(\beta)(x) = \beta f(x)/(1 + \beta g(x))$: here f and g are homogeneous polynomials in R^n of degree 4 and 2 respectively which are positive as $x \neq 0$. With regard to the proofs we remark that we have obtained the convergence of the eigenprojections as $\beta \to 0$ exploiting only the strong convergence of the resolvents of $H_0 + V(\beta)$ thanks to the results of Vock and Hunziker [5], while the proof of the Borel summability is obtained by a Watson-like theorem [4: th. XII. 21] and the results of [2]; these proofs can be trivially extended to a more general potential $V(\beta)$ analytic with respect to β in a sector $\mathscr{A} = \{\beta \in C ; |\arg \beta| < \vartheta\}$ (with $\vartheta \in (\pi/2, \pi)$), real valued for $\beta \in R^+$ and satisfying the following conditions:

i) $V(\beta)(x) \underset{\beta \to 0}{\to} 0 \qquad \forall x \in R^n$

ii) $\forall \beta \in \mathscr{A} \qquad \exists C(\beta) > 0$ such that

$$V(\beta)(x) \leq C(\beta)|x|^2$$

$$\operatorname{Re} V(\beta)(x) \geq -C(\beta) \qquad \forall x \in R^n \qquad \forall \beta \in \mathscr{A}$$

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iii) $\arg V(\beta)(x) \in [0, \vartheta]$ if $\arg \beta \in [0, \vartheta]$ $\forall x \in R^n$

$\arg V(\beta)(x) \in [-\vartheta, 0]$ if $\arg \beta \in [-\vartheta, 0]$ $\forall x \in R^n$

iv) the following expansion holds for every $N = 1, 2, 3, \cdots$

$$V(\beta)(x) = \sum_{j=0}^{N} a_j(x)\,\beta^j + \beta^{N+1} R_{N+1}(\beta)$$

where

$$|a_j(x)| \leq c\,|x|^{2j+2} \qquad \forall x \in R^n \qquad j = 0, 1, 2, \cdots$$

and

$$|R_{N+1}(\beta)| \leq c\,|x|^{2N+4} \qquad \forall x \in R^n .$$

Before stating our results we recall some definitions. An operator-valued function $H(\beta)$ defined in a complex domain Ω is said to be a holomorphic family of type A (in the Hilbert space $\mathscr{H}$) [3: ch. VII § 2.1] if

i) $H(\beta)$ is a closed operator in $\mathscr{H}$ with domain $D(H(\beta)) = D$ independent of $\beta \in \Omega$,

ii) $\beta \rightarrow H(\beta)\,u$ is a (vector-valued) holomorphic function in Ω for every $u \in D$.

The family $H(\beta)$ is said to be selfadjoint [3: ch. VII § 3.1] if Ω is symmetric with respect to the real axis, D is dense in $\mathscr{H}$ and $H(\beta)^* = H(\bar{\beta})$ for every $\beta \in \Omega$; as for the spectral properties of these families we refer to [3: ch. VII § 1.3, 1.5, 3.2]; here we recall only that if λ_0 is an isolated nondegenerate eigenvalue of $H(0)$ (suppose that $0 \in \Omega$), then there exists an analytic function $\lambda(\beta)$ defined for $|\beta| < \delta$ such that $\lambda(0) = \lambda_0$ and $\lambda(\beta)$ is a nondegenerate eigenvalue of $H(\beta)$ for every β with $|\beta| < \delta$ [4: th. XII. 3]. The Taylor series of $\lambda(\beta)$ is called Rayleigh-Schrödinger series and its coefficients can be computed expanding the right hand side of

$$(*) \qquad \lambda(\beta) = \frac{\langle \Omega(0), H(\beta)\,P(\beta)\,\Omega(0)\rangle}{\langle \Omega(0), P(\beta)\,\Omega(0)\rangle} ;$$

here

$$(**) \qquad P(\beta) = -(2\pi i)^{-1} \int_{|z-\lambda_0|=r} (H(\beta) - z)^{-1}\,dz$$

denotes the eigenprojection onto the eigenspace relative to $\lambda(\beta)$ and $\Omega(0)$ the eigenvector of $H(0)$ relative to λ_0.

We say that an operator-valued function $H(\beta)$ defined in a (real or complex) domain Ω converges strongly in the generalized sense to H_0 as $\beta \rightarrow \beta_0$ ($\beta_0 \in \bar{\Omega}$) [3: ch. VIII § 1.1] if there exists $z \in C$ such that the resolvents $R(\beta, z) = (H(\beta) - z)^{-1}$ of $H(\beta)$ converge strongly to the resolvent $R_0(z) = (H_0 - z)^{-1}$

of H_0 as $\beta \to \beta_0$; in these hypotheses an isolated eigenvalue λ_0 of H_0 with finite multiplicity is said to be stable [3: ch. VIII § 1.4] if

i) there exists $\delta > 0$ such that for every z with $0 < |z - \lambda_0| < \delta$ there exists $R(\beta, z)$ for every β close to β_0 and $R(\beta, z)$ converges strongly to $R_0(z)$ as $\beta \to \beta_0$,

ii) if $P(\beta)$ denotes the projection given by (**) $(0 < r < \delta)$ and P_0 denotes the eigenprojection for the eigenvalue λ_0 of H_0 then $\dim P(\beta) \leq \dim P_0$ for β sufficiently close to β_0.

In what follows we shall denote by $H(\beta)$ the operator $-\Delta + |x|^2 + \beta f(x)/(1+\beta g(x))$ with domain $D(H(\beta)) = \{f \in W^2(R^n); (1+|x|^2) f \in L^2(R^n)\}$; we collect our results on $H(\beta)$ in the following theorem.

THEOREM 1. *The operator family* $H(\beta)$ $\beta \in C \setminus (-\infty, 0]$ *is selfadjoint holomorphic of type* A; *it consists of operators with compact resolvents and discrete spectra and converges strongly in the generalized sense to* $H(0)$ *as* $\beta \to 0$ *and* $|\arg \beta| < \pi - \delta$ $(0 < \delta < \pi/2)$; *its domain of boundedness is* $\Delta_b = C \setminus \sigma(H(0))$ *and every eigenvalue of* $H(0)$ *is stable.*

It is well known that the least eigenvalue of $H(0)$ is n and is nondegenerate; expanding the right hand side of (*) we obtain the coefficients of the so called perturbation series of $\lambda(0) = n$; this series may result divergent for every $\beta \neq 0$ since the family $H(\beta)$ is not holomorphic for $\beta = 0$. Nevertheless the following theorem provides a connection between the perturbation series and the eigenvalue $\lambda(\beta)$ of $H(\beta)$ which converges to $\lambda(0) = n$ as $\beta \to 0$. Referring the reader to [4: ch. XII § 4] for the definition of the Borel summability, we have

THEOREM 2. *There exists* $r > 0$ *such that the Rayleigh–Schrödinger perturbation series of the least eigenvalue of* $H(0)$ *is Borel summable to* $\lambda(\beta)$ *for every* $\beta \in C$ *with* $|\beta| < r$ *and* $|\arg \beta| < \delta$.

REFERENCES

[1] G. AUBERSON (1982) – « Commun. Math. Phys. », *84*, 53.
[2] G. AUBERSON and G. MENNESSIER (1981) – « J. Math. Phys. » *22*, 2472.
[3] T. KATO (1975) – *Perturbation Theory for Linear Operators*. Springer–Verlag, Berlin–Heidelberg–New York.
[4] M. REED and B. SIMON (1978) – *Analysis of Operators*. Acad. Press. New York.
[5] E. VOCK and W. HUNZIKER (1982) – « Commun. Math. Phys », *83*, 208.