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On time-varying networks of time-varying semiautomata

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Scienza dell'informazione. — *On time-varying networks of time-varying semiautomata* (*), Nota di LOUISE MARTIN e DAN A. SIMOVICI, presentata (**) dal Socio E. MARTINELLI.

RIASSUNTO. — Si utilizza la nozione di réte di semiautòmi con struttura variabile nel tempo, per ottenere un criterio di decomposizione dei semiautòmi con struttura variabile. Si mette in evidenza il ruolo di una congruenza nella decomposizione di questo tipo di semiautònomi.

1. INTRODUCTION

A time-vaying semi-automaton (tvsa) is a 3-uple $\mathscr{A} = (\mathrm{I}, \mathrm{S}, \{f_n \mid n \in \mathbf{N}\})$, where I, S are finite sets representing, respectively the set of inputs and the set of states and $f_n : \mathrm{S} \times \mathrm{I} \to \mathrm{S}$ is the next-state function at time n.

By I^* we shall denote the free monoid generated by the set I. In the sequel, $l(p)$ will be the length of the word $p \in \mathrm{I}^*$ and e will be the empty work of this monoid.

We shall extend the next-state function to $\mathrm{I}^* \times \mathrm{S}$ by induction on the length of words. Namely, we shall have

$$f_n(s, e) = s \quad \text{and} \quad f_n(s, pi) = f_{n+l(p)}(f_n(s, p), i)),$$

for every $n \in \mathrm{N}$, $s \in \mathrm{S}$, $p \in \mathrm{I}^*$ and $i \in \mathrm{I}$.

It is easy to see that

$$f_n(s, rp) = l_{n+l(r)}(f_n(s, r), p),$$

for $s \in \mathrm{S}$, $r, p \in \mathrm{I}^*$.

The other main object of the note is the time-variant network of tvsa (tvnsa). An analog concept was considered in [2].

DEFINITION. A tvnsa is a 3-uple.

$$\mathscr{N} = (\{\mathscr{A}_j \mid 1 \leq j \leq k\}, \mathrm{I}, \{\chi_j^t \mid t \in \mathrm{N}, 1 \leq j \leq k\}),$$

where $\mathscr{A}_j = (\mathrm{I}_j, \mathrm{S}_j, \{\varphi_j^n \mid n \in \mathbf{N}\})$ are the component tvsa of the network and χ_j^t is the j-th transmission function at time t,

$$\chi_j^t : \Pi\{\mathrm{S}_l \mid 1 \leq l \leq k\} \times \mathrm{I} \to \mathrm{I}_j.$$

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A tvnsa can be regarded as a special tvsa

$$\mathscr{A}_{\mathscr{N}} = (\mathrm{I}\,,\, \Pi\,\{\mathrm{S}_l \mid 1 \leq l \leq k\}\,,\, \{\mathrm{F}_n \mid n \in \mathbf{N}\})\,,$$

where

$$\mathrm{F}_n\,((s_1\,,\cdots,\,s_k)\,,\,i) = (\varphi_1^n\,(s_1\,,\,\chi_1^n\,(s_1\,,\cdots,\,s_k\,,\,i)\,,\cdots,$$
$$\varphi_k^n\,(s_k\,,\,\chi_k^n\,(s_1\,,\cdots,\,s_k\,,\,i)))\,.$$

A tvnsa is in the *normal form if for each component* tvsa $\mathscr{A}_j$ there exist k sets $\mathrm{S}_j^1\,,\cdots,\,\mathrm{S}_j^k$ such that

$$\mathrm{I}_j = \prod_{l=1}^{k} \mathrm{S}_j^l \times \mathrm{I}$$

and for each transmission function χ_j^t there exist k component transmission functions $\chi_{lj}^t : \mathrm{S}_l \rightarrow \mathrm{S}_j^l\,,\ 1 \leq j\,,\, l \leq k,$ such that

$$\chi_j^t\,(s_1\,,\cdots,\,s_k\,,\,i) = (\chi_{1j}^t\,(s_1)\,,\cdots,\,\chi_{kj}^t\,(s_k)\,,\,i)\,,$$

for $(s_1\,,\cdots,\,s_k) \in \Pi\,\{\mathrm{S}_l \mid 1 \leq l \leq k\}$ and $i \in \mathrm{I}$.

2. SIMULATION OF TIME-VARYING SEMI-AUTOMATA BY NETWORKS

Let Eq (S) be the set of equivalences on a set S. We shall denote by ι_S and ω_S, the diagonal relation and the total relation, respectively. Eq (S) is a lattice under the operation of set-intersection of equivalences " $\bigcap$ " and the union of equivalence " $\vee$ ".

An association algebra (see [1]) is a sublattice $\alpha \subseteq \mathrm{Eq}\,(\mathrm{S}) \times \mathrm{Eq}\,(\mathrm{S})$ such that $(\iota_\mathrm{S}\,,\,\sigma)\,,\,(\sigma\,,\,\omega_\mathrm{S}) \in \alpha$, for every $\sigma \in \mathrm{Eq}\,(\mathrm{S})$.

Let $\mathscr{A} = (\mathrm{I}\,,\,\mathrm{S}\,,\,\{f_n \mid n \in \mathbf{N}\})$ be a tvsa. It is known (see [4]) that the set $\alpha_n = \{(\sigma\,,\,\tau) \mid (s', s'') \in \sigma$ implies $(f_n\,(s', i)\,,\, f_n\,(s'', i)) \in \tau\,,\ \forall i \in \mathrm{I}\}$ is an association algebra. We shall refer to α_n as the association algebra of $\mathscr{A}$ at time n.

A sequence $\mathrm{H} = (h^0,\cdots,\,h^n,\cdots)$ is a tvsa homomorphism from the semi-automaton $\mathscr{A} = (\mathrm{I}\,,\,\mathrm{S}\,,\,\{f_n \mid n \in \mathbf{N}\})$ to the semi-automaton

$$\mathscr{A}' = (\mathrm{I}\,,\,\mathrm{S}',\,\{f_n' \mid n \in \mathbf{N}\}) \qquad \text{if} \quad h^n : \mathrm{S} \rightarrow \mathrm{S}'$$

for $n \in \mathbf{N}$ and $h_{n+1}\,(f_n\,(s\,,\,i)) = f_n'\,(h_n\,(s)\,,\,i)$ for $n \in \mathbf{N}\,,\ s \in \mathrm{S}\,,\ i \in \mathrm{I}$. Therefore, if there exists such a homomorphism between $\mathscr{A}$ and $\mathscr{A}'$ we can write $f_n\,(s\,,\,i) \in h_{n+1}^{-1}\,(f_n'\,(h_n\,(s)\,,\,i))$, which indicates that $\mathscr{A}'$ is able to simulate $\mathscr{A}$.

Let $\mathscr{N}$ be a network of tvsa. If there is a homomorphism from the tvsa $\mathscr{A}$ to the tvsa $\mathscr{A}_{\mathscr{N}}$ we shall say that $\mathscr{N}$ simulates the semiautomaton $\mathscr{A}$.

THEOREM 1. *The tvsa* $\mathscr{A} = (\mathrm{I}\,,\,\mathrm{S}\,,\,\{f_n \mid n \in \mathbf{N}\})$ *is simulated by the tvnsa* $\mathscr{N}$ *(which is the normal form and contains k tvsa) if and only if* *there exist two families of equivalences on* $S : \{\sigma_l^n \mid n \in \mathbf{N}, 1 \leq l \leq k\}$ *and* $\{\sigma_{jl}^n \mid n \in \mathbf{N}, 1 \leq j, l \leq k\}$ *satisfying the following conditions*:

i) $\sigma_l^n \subseteq \sigma_{lj}^n$ and

ii) $(\sigma_l^n \cap \bigcap \{\sigma_{jl}^n \mid 1 \leq j \leq k\}, \sigma_l^{n+1}) \in \alpha_n$,

where α_n *is the association algebra of* $\mathscr{A}$ *at time n, for* $1 \leq j, l \leq k$ *and* $n \in \mathbf{N}$.

Proof. Let $\mathscr{N} = (\{\mathscr{A}_j \mid 1 \leq j \leq k\}, I, \{\chi_j^t \mid t \in N, 1 \leq j \leq k\})$ be a tvnsa in standard form, which contains k semi-automata $\mathscr{A}_j = \left(\prod_{l=1}^{k} S_j^l \times I, S_j, \{\varphi_j^n \mid n \in \mathbf{N}\}\right)$ for $1 \leq j \leq k$. Assume that there exists a homomorphism $H = (h^0, \cdots, h^n, \cdots) : \mathscr{A} \rightarrow \mathscr{A}_{\mathscr{N}}$. If $h^n(s) = (s_1, \cdots, s_k)$ then $s_j = pr_j h^n(s)$ where $pr_j : \prod_{l=1}^{k} S_l \rightarrow S_j$ gives the j^{th} projection of the k-tuples. The homomorphism condition can be now written:

$$(pr_1 h^{n+1}(f_n(s, i)), \cdots, pr_k h^{n+1}(f_n(s, i))) =$$
$$= (\varphi_1^n(s_1, \chi_{11}^n(pr_1 h^n(s)), \cdots, \chi_{k1}^n(pr_k h^n(s), i), \cdots$$
$$\cdots, \varphi_k^n(s_k, \chi_{1k}^n(pr_1 h^n(s)), \cdots, \chi_{kk}^n(pr_k h^n(s), i)),$$

hence

$$(1) \qquad pr_l h^{n+1}(f_n(s, i)) = \varphi_l^n(pr_l h^n(s), \chi_{1l}^n(pr_1 h^n(s)), \cdots, \chi_{kl}^n(pr_k h^n(s)), i),$$

for $1 \leq l \leq k$.

Let us define now the equivalences σ_p^n and σ_{pq}^n on S by $\sigma_p^n = \ker pr_p h^n$ and $\sigma_{pq}^n = \ker(\chi_{pq}^n pr_p h^n)$ for $1 \leq p, q \leq k, n \in \mathbf{N}$. It is obvious that $\sigma_l^n \subseteq \sigma_{lj}^n$ for $1 \leq j, l \leq k, n \in \mathbf{N}$.

In view of the equality (1), if $(s', s'') \in \sigma_l^n \cap \bigcap \{\sigma_{jl} \mid 1 \leq j \leq k\}$, it follows that $(f_n(s', i), f_n(s'', i)) \in \sigma_l^{n+1}$, which gives the second condition of the Theorem.

Conversely, the existence of the previous equivalence guarantees not only the existence of $\mathscr{N}$, but implies that the transmission functions are constant in time.

Indeed, assume that we have the families $\{\sigma_l^n \mid n \in N, 1 \leq l \leq k\}$ and $\{\sigma_{jl}^n \mid n \in \mathbf{N}, 1 \leq j, l \leq k\}$ satisfying the conditions of the Theorem and let us consider the equivalences $\rho_l = \bigcap \{\sigma_l^n \mid n \in \mathbf{N}\}$ and $\rho_{lj} = \bigcap \{\sigma_{lj}^n \mid n \in \mathbf{N}\}$. In view of the first condition we have $\rho_l \subseteq \rho_{lj}$. We shall consider the tvsa $\mathscr{A}_p = \left(\mathscr{P}\left(\prod_{q=1}^{k} S/\rho_{qp} \times I\right), \mathscr{P}(S/\rho_p), \{\varphi_p^n \mid n \in \mathbf{N}\}\right)$ for $1 \leq p \leq k$ and the tvnsa in normal form defined by the transmission functions

$$\chi_{pq}^n : \mathscr{P}(S/\rho_p) \rightarrow \mathscr{P}(S/\rho_{pq})$$

given by

$$\chi_{pq}^{n}(\{[s]_{\rho_p} \mid s \in S_1\}) = \{[s]_{\rho_{pq}} \mid s \in S_1\} ,$$

for all $s \in S_1$. Since $\rho_p \subseteq \rho_{pq}$ this definition is a correct one. It is clear that χ_{pq}^{n} do not depend effectively on the time n.

The mapping h^n is defined by

$$h^n(s) = (\{[t]_{\rho_1} \mid (t, s) \in \sigma_1^n\}, \cdots, \{[t]_{\rho_k} \mid (t, s) \in \sigma_k^n\}) ,$$

for $s \in S$.

The condition (1) is in this case

$$\{[t]_{\rho_l} \mid (t, f_n(s, i)) \in \sigma_l^{n+1}\} = \varphi_l^n(\{[t]_{\rho_l} \mid (t, s) \in \sigma_l^n\} ,$$
$$(\{[t]_{\rho_{1l}} \mid (t, s) \in \sigma_1^n\}, \cdots, \{[t]_{\rho_{kl}} \mid (t, s) \in \sigma_k^n\}, i)) ,$$

and we can satisfy it by defining the transition functions of the components by:

$$\varphi_l^n(\{[t]_{\rho_l} \mid (t, s) \in \sigma_l^n\}, (\{[t]_{\rho_{1l}} \mid (t, s_1) \in \sigma_1^n, \cdots, \{[t]_{\rho_{kl}} \mid (t, s_k) \in \sigma_k^n\}, i)) =$$

$$= \begin{cases} \{[t]_{\rho_l} \mid (t, f_n(s, i)) \in \sigma_l^{n+1}\} & , \quad \text{if} \quad s_1 = \cdots = s_k = s , \\ \text{arbitrary, otherwise.} \end{cases}$$

This proves the sufficiency of the conditions of this Theorem. ***

3. Serial and parallel decompositions of time-varying semi-automata

Let $\mathcal{N}$ be a tvnsa in normal form which contains two tvsa

$\mathcal{A}_1 = (S_1^1 \times S_1^2 \times I, S_1, \{\varphi_1^n \mid n \in \mathbf{N}\})$ and $\mathcal{A}_2 = (S_2^1 \times S_2^2 \times I, S_2, \{\varphi_2^n \mid n \in \mathbf{N}\})$.

A tvsa $\mathcal{A} = (I, S, \{\varphi^n \mid n \in \mathbf{N}\})$ can be simulated by $\mathcal{N}$ if, according to Theorem 1, we have the families of equivalences $\{\sigma_1^n \mid n \in \mathbf{N}\}$, $\{\sigma_2^n \mid \in \mathbf{N}\}$ and $\{\sigma_{jl}^n \mid n \in \mathbf{N}, j, l \in \{1, 2\}\}$ such that

i) $\sigma_1^n \subseteq \sigma_{11}^n$, $\sigma_1^n \subseteq \sigma_{12}^n$,

ii) $\sigma_2^n \subseteq \sigma_{21}^n$ $\sigma_2^n \subseteq \sigma_{22}^n$,

iii) $(\sigma_1^n \cap \sigma_{21}^n, \sigma_1^{n+1}) \in \alpha_n$ and

iv) $(\sigma_2^n \cap \sigma_{12}^n, \sigma_2^{n+1}) \in \alpha_n$,

for $n \in \mathbf{N}$.

By defining a new sequence of equivalences on S by $\theta^n = \sigma_1^n \cap \sigma_2^n$ for $n \in \mathbf{N}$, we obtain $\theta^n \subseteq \sigma_{11}^n \cap \sigma_{12}^n \cap \sigma_{21}^n \cap \sigma_{22}^n$, for $n \in \mathbf{N}$. Since α_n is a sublattice of Eq (S) $\times$ Eq (S) we shall have:

$$(\sigma_1^n \cap \sigma_2^n \cap \sigma_{21}^n \cap \sigma_{12}^n , \sigma_1^{n+1} \cap \sigma_2^{n+1}) \in \alpha_n ,$$

or

$$(\theta^n, \theta^{n+1}) \in \alpha_n ,$$

for $n \in \mathbf{N}$, which points that the sequence $\theta = (\theta^0, \cdots, \theta^n, \cdots)$ is a tvsa congruence, in the sense of [4].

Conversely, if there exists a tvsa congruence θ on $\mathscr{A}$, by choosing $\sigma_1^n = \sigma_2^n = \sigma_{11}^n = \sigma_{12}^n = \sigma_{21}^n = \sigma_{22}^n = \theta^n$ for $n \in \mathbf{N}$ we can clearly satisfy the above conditions. Using the Corollary from [4] and our Theorem 1, we obtain.

THEOREM 2. *The following four assertions are equivalent for a tvsa* $\mathscr{A}$:

i) there exists a tvsa congruence of $\mathscr{A}$;

ii) $\mathscr{A}$ can be simulated by a two semi-automata network in normal form;

iii) there exists a serial decomposition of $\mathscr{A}$ and

ii) there exists a monitorial decomposition of $\mathscr{A}$.

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