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On locally S-closed spaces

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Topologia. — *On locally S-closed spaces.* Nota di TAKASHI NOIRI, presentata (*) dal Socio E. MARTINELLI.

RIASSUNTO. — Si studiano le condizioni sotto cui l'immagine (o l'immagine inversa) di uno spazio localmente S-chiuso sia localmente S–chiuso.

1. INTRODUCTION

In 1976, Thompson [16] introduced the concept of S-closed spaces. In 1978, the present author [9] introduced the concept of locally S-closed spaces and showed that locally S-closed spaces are preserved under open continuous surjections. Recently, Mashhour and Hasanein [7] have shown that the condition " continuous " in the above result may be replaced by " almost-continuous " in the sense of Singal [15]. The purpose of the present paper is to investigate some conditions on functions for the images (inverse images) of locally S-closed spaces to be locally S-closed. It will be shown in Section 3 that locally S-closed spaces are preserved under weakly-open and θ-continuous surjections and that a topological space X is locally S-closed if and only if the semi-regularization X^* is locally S-closed. In the last section, we shall show that locally S-closed spaces are inverse-preserved under *s*-perfect almost-continuous (in the sense of Husain [4]) surjections.

2. PRELIMINARIES

Throughout the present paper X and Y represent topological spaces on which no separation axioms are assumed unless explicitly stated. By $f: X \to Y$ we denote a function on which the continuity is not assumed. Let S be a subset of X. The closure of S and the interior of S in X are denoted by $\mathrm{Cl}_X(S)$ and $\mathrm{Int}_X(S)$ (or simply Cl (S) and Int (S)), respectively. A subset S of X is said to be *semi-open* [6] if there exists an open set U of X such that $U \subset S \subset \mathrm{Cl}(U)$. A subset S is said to be *regular open* (resp. *regular closed*) if $\mathrm{Int}(\mathrm{Cl}(S)) = S$ (resp. $\mathrm{Cl}(\mathrm{Int}(S)) = S$). A space X is said to be *extremely disconnected* if the closure of every open set of X is open in X.

DEFINITION 2.1. A subset K of X is said to be S-*closed relative to* X [9] if for every cover $\{U_\alpha \mid \alpha \in \nabla\}$ of K by semi-open sets of X, there exists a finite subset ∇_0 of ∇ such that $K \subset \bigcup \{\mathrm{Cl}_X(U_\alpha) \mid \alpha \in \nabla_0\}$. If the set X is S-closed relative to X, then the space X is called S-*closed* [16].

(*) Nella seduta del 12 febbraio 1983.

LEMMA 2.2. *A subset* K *of* X *is* S-*closed relative to* X *if and only if every cover of* K *by regular closed sets of* X *has a finite subcover.*

Proof. Since every regular closed set is semi-open, the proof is obvious and is thus omitted.

DEFINITION 2.3. A space X is said to be *locally* S-*closed* [9] if each point of X has an open neighborhood which is an S-closed subspace of X.

LEMMA 2.4 (NOIRI [9]). *For a space* X *the following are equivalent*:

(1) X *is locally* S-*closed.*

(2) *Each point of* X *has an open neighbourhood which is* S-*closed relative to* X.

(3) *Each point of* X *has an open neighbourhood* U *such that* $Cl_X(U)$ *is* S-*closed relative to* X.

We shall recall some definitions of functions used in this paper.

DEFINITION 2.5. A function $f: X \to Y$ is said to be *almost-continuous* [15], briefly a.c.S., (resp θ-*continuous* [3], *weakly-continuous* [5]) if for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U containing x such that $f(U) \subset Int(Cl(V))$ (resp. $f(Cl(U)) \subset Cl(V)$, $f(U) \subset Cl(V)$).

DEFINITION 2.6. A function $f: X \to Y$ is said to be *almost-continuous* [4], briefly a.c.H., if for each $x \in X$ and each open set V containing $f(x)$, $Cl_X(f^{-1}(V))$ is a neighbourhood of x.

DEFINITION 2.7. A function $f: X \to Y$ is said to be *semi-continuous* [6] if for every open set V of Y, $f^{-1}(V)$ is semi-open in X.

The following implications are well-known: continuous $\Rightarrow$ a.c.S. $\Rightarrow$ θ-continuous $\Rightarrow$ weakly-continuous. It is known that " semi-continuity " and " weak-continuity " are independent of each other [13, p. 318] and moreover " a.c.H. " and " weakly-continuous " are also independent of each other [14, Example 1].

DEFINITION 2.8. A function $f: X \to Y$ is said to be *weakly-open* [14] (resp. *semi-open* [1]) if for every open set U of X, $f(U) \subset Int(f(Cl(U)))$ (resp. $f(U) \subset Cl(Int(f(U)))$).

DEFINITION 2.9. A function $f: X \to Y$ is said to be *almost-open* [15], briefly a.o.S., if for every regular open set U of X, $f(U)$ is open in Y.

DEFINITION 2.10. A function $f: X \to Y$ is said to be *almost-open* [17], briefly a.o.W., if for every open set V of Y, $f^{-1}(Cl(V)) \subset Cl(f^{-1}(V))$, where f is not always injective.

The relationships among these weak forms of openness are investigated in [13].

3. The images of locally S-closed spaces

Lemma 3.1 (Rose [14]). *A function $f: X \to Y$ is a.o.W. if and only if for every open set* U *of* X, $f(U) \subset \mathrm{Int}(\mathrm{Cl}(f(U)))$.

Theorem 3.2. *If $f: X \to Y$ is a weakly-continuous a.o.W. surjection and* X *is locally* S*-closed, then* Y *is locally* S*-closed.*

Proof. Let y be any point of Y. There exists $x \in X$ such that $f(x) = y$. Since X is locally S-closed, by Lemma 2.4 there exists an open neighborhood U of x which is S-closed relative to X. Since f is a.o.W., by Lemma 3.1 we have $y \in f(U) \subset \mathrm{Int}(\mathrm{Cl}(f(U)))$. It follows from Theorem 4.5 of [10] that $f(U)$ is S-closed relative to Y and hence so is $\mathrm{Int}(\mathrm{Cl}(f(U)))$ [9, Theorem 3.4]. This shows that Y is locally S-closed.

Corollary 3.3. *Let $f: X \to Y$ be a semi-continuous a.o.W. surjection. If* X *is locally* S*-closed Hausdorff, then* Y *is locally* S*-closed.*

Proof. Since X is locally S-closed Hausdorff, by Theorem 3.2 of [11] X is extremely disconnected and hence f is weakly-continuous [13, Theorem 3.2]. Therefore, it follows from Theorem 3.2 that Y is locally S-closed.

Corollary 3.4. *Let $f: X \to Y$ be a semi-continuous a.o.S. surjection. If* X *is locally* S*-closed Hausdorff, then* Y *is locally* S*-closed.*

Proof. By Corollary 3.3, it is sufficient to show that f is a.o.W. if it is semi-continuous a.o.S. Let V be an open set of Y. Since f is semi-continuous, $f^{-1}(V)$ is semi-open in X and hence $f^{-1}(V) \subset \mathrm{Cl}(\mathrm{Int}(f^{-1}(V)))$. Now, put

$$F = Y - f(X - \mathrm{Cl}(\mathrm{Int}(f^{-1}(V)))) .$$

Then F is closed in Y because f is a.o.S. Furthermore, we have $V \subset F$ and $f^{-1}(F) \subset \mathrm{Cl}(\mathrm{Int}(f^{-1}(V)))$. Therefore, we have $f^{-1}(\mathrm{Cl}(V)) \subset \mathrm{Cl}(f^{-1}(V))$.

Lemma 3.5. *Let $f: X \to Y$ be a θ-continuous weakly-open function. If* K *is* S*-closed relative to* X, *then* $f(K)$ *is* S*-closed relative to* Y.

Proof. Let $\{V_\alpha \mid \alpha \in \nabla\}$ be a cover of $f(K)$ by regular closed sets of Y. It follows from Theorem 4.4 of [12] that $\{f^{-1}(V_\alpha) \mid \alpha \in \nabla\}$ is a cover of K by regular closed sets of X. Since K is S-closed relative to X, by Lemma 2.2 there exists a finite subset ∇_0 of ∇ such that $K \subset \bigcup \{f^{-1}(V_\alpha) \mid \alpha \in \nabla_0\}$. Therefore, we have $f(K) \subset \bigcup \{V_\alpha \mid \alpha \in \nabla_0\}$ and hence by Lemma 2.2 $f(K)$ is S-closed relative to Y.

Theorem 3.6. *If $f: X \to Y$ is a θ-continuous weakly-open surjection and* X *is locally* S*-closed, then* Y *is locally* S*-closed.*

Proof. Let y be a point of Y. There exists $x \in X$ such that $f(x) = y$. Since X is locally S-closed, by Lemma 2.4 there exists an open neighborhood U of x such that Cl (U) is S-closed relative to X. Since f is weakly-open, we have $y \in f(U) \subset \mathrm{Int}(f(\mathrm{Cl}(U)))$. By Lemma 3.5, $f(\mathrm{Cl}(U))$ is S-closed relative to Y and hence so is $\mathrm{Int}(\mathrm{Cl}(f(\mathrm{Cl}(U))))$ [9, Theorem 3.4]. Therefore, it follows from Lemma 2.4 that Y is locally S-closed.

COROLLARY 3.7. *If $f: X \to Y$ is an a.c.S. a.o.S. surjection and* X *is locally* S-*closed, then* Y *is locally* S-*closed.*

Proof. Every a.c.S. function is θ-continuous and every a.o.S. function is weakly-open [13, Lemma 1.4]. Therefore, this is an immediate consequence of Theorem 3.6.

COROLLARY 3.8 (MASHHOUR and HASANEIN [7]). *The local* S-*closedness is preserved under a.c.S. open surjections.*

COROLLARY 3.9. *A space* X *is locally* S-*closed if and only if the semi-regularization* X^* *is locally* S-*closed.*

Proof. The identity function $i_X : X \to X^*$ and the inverse $(i_X)^{-1}$ are a.c.S. a.o.S. Therefore, this is an immediate consequence of Corollary 3.7.

4. THE INVERSE IMAGES OF LOCALLY S-CLOSED SPACES

For a subset A of X, the *s-closure* of A [2], denoted by $\mathrm{cl}_s(A)$, is defined as follows: $\{x \in X \mid \mathrm{Cl}(U) \cap A \neq \varnothing$ for every semi-open set U containing $x\}$. A function $f: X \to Y$ is said to be *s-closed* if for every subset A of X $\mathrm{cl}_s(f(A)) \subset f(\mathrm{cl}_s(A))$.

DEFINITION 4.1. A function $f: X \to Y$ is said to be *s-perfect* [2] if f is s-closed and point inverses are S-closed relative to X.

The following lemma of Dickman and Krystock [2, Proposition 3.3] is true without assuming Hausdorffness on X and Y.

LEMMA 4.2 (DICKMAN and KRYSTOCK [2]). *If $f: X \to Y$ is s-perfect and* K *is* S-*closed relative to* Y, *then $f^{-1}(K)$ is* S-*closed relative to* X.

THEOREM 4.3. *If $f: X \to Y$ is an a.c.H. s-perfect surjection and* Y *is locally* S-*closed, then* X *is locally* S-*closed.*

Proof. For any point $x \in X$, we put $y = f(x)$. Since Y is locally S-closed, by Lemma 2.4 there exists an open neighbourhood V of $f(x)$ which is S-closed relative to Y. Since f is a.c.H., we have $x \in f^{-1}(V) \subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))$. By Lemma 4.2, $f^{-1}(V)$ is S-closed relative to X and hence so is $\mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))$ [9, Theorem 3.4]. It follows from Lemma 2.4 that X is locally S-closed.

LEMMA 4.4. *Let* Y *be extremely disconnected and* $f : X \to Y$ *be a semi-open surjection with* $f^{-1}(f(U)) \subset \mathrm{Cl}_X(U)$ *for every semi-open set* U *of* X. *If* f *is a.c.H.* (*or weakly-continuous*) *and* K *is* S-*closed relative to* Y, *then* $f^{-1}(K)$ *is* S-*closed relative to* X.

Proof. Let $\{U_\alpha \mid \alpha \in \nabla\}$ be a cover of $f^{-1}(K)$ by semi-open sets of X. Since f is a.c.H. (resp. weakly-continuous), by Theorem 2.5 (resp. Corollary 2.4) of [13] f is pre-semi-open and hence $\{f(U_\alpha) \mid \alpha \in \nabla\}$ is a cover of K by semi-open sets of Y. Since K is S-closed relative to Y, there exists a finite subset ∇_0 of ∇ such that

$$K \subset \bigcup \{\mathrm{Cl}_Y(f(U_\alpha)) \mid \alpha \in \nabla_0\}.$$

By using [8, Theorem 2] and [13, Lemma 1.13], we have

$$f^{-1}(K) \subset \bigcup_{\alpha \in \nabla_0} f^{-1}(\mathrm{Cl}_Y(f(U_\alpha))) \subset \bigcup_{\alpha \in \nabla_0} \mathrm{Cl}_X(f^{-1}(f(U_\alpha))) \subset \bigcup_{\alpha \in \nabla_0} \mathrm{Cl}_X(U_\alpha).$$

This shows that $f^{-1}(K)$ is S-closed relative to X.

THEOREM 4.5. *Let* $f : X \to Y$ *be a semi-open surjection with* $f^{-1}(f(U)) \subset \mathrm{Cl}_X(U)$ *for every semi-open set* U *of* X. *If* f *is a.c.H.* (*or weakly-continuous*) *and* Y *is locally* S-*closed Hausdorff, then* X *is locally* S-*closed.*

Proof. (*i*) Let f be a.c.H. For a point $x \in X$, by Lemma 2.4 there exists an open neighborhood V of $f(x)$ which is S-closed relative to Y. Since Y is locally S-closed Hausdorff, by Theorem 3.2 of [11] Y is extremely disconnected and hence, by Lemma 4.4, $f^{-1}(V)$ is S-closed relative to X. Moreover, $\mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))$ is S-closed relative to X [9, Theorem 3.4]. Since f is a.c.H., we have $x \in f^{-1}(V) \subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))$. It follows from Lemma 2,4 that X is locally S-closed.

(*ii*) Let f be weakly-continuous. For a point $x \in X$, there exists an open neighbourhood V of $f(x)$ such that Cl (V) is S-closed relative to Y. By Lemma 4.4, $f^{-1}(\mathrm{Cl}(V))$ is S-closed relative to X. Since f is weakly-continuous, by Theorem 1 of [5] we have

$$x \in f^{-1}(V) \subset \mathrm{Int}(f^{-1}(\mathrm{Cl}(V))) \subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(\mathrm{Cl}(V)))).$$

It follows from Theorem 3.4 of [9] and Lemma 2.4 that X is locally S-closed.

COROLLARY 4.6. *Let* $f : X \to Y$ *be a semi-open weakly-continuous bijection. If* Y *is locally* S-*closed Hausdorff, then so is* X.

Proof. It follows from Theorem 4.5 that X is locally S-closed. Since Y is locally S-closed Hausdorff, it is extremely disconnected [11, Theorem 3.2]. Let x and y be any distinct points of X. There exist open sets U and V of Y such that $f(x) \in U$, $f(y) \in V$ and $U \cap V = \varnothing$; hence $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \varnothing$.

Since f is weakly-continuous, Int $(f^{-1}(\mathrm{Cl}(\mathrm{U})))$ and Int $(f^{-1}(\mathrm{Cl}(\mathrm{V})))$ are disjoint open neighbourhoods of x and y, respectively [5, Theorem 1]. Therefore, X is Hausdorff.

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