Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Edoardo Vesentini

Non-subharmonicity of the Hausdorff distance

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **74** (1983), n.2, p. 62–65.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1983_8_74_2_62_0>

Geometria. — *Non-subharmonicity of the Hausdorff distance.* Nota (*) del Corrisp. EDOARDO VESENTINI.

RIASSUNTO. — Si dimostra con esempi che la distanza di Hausdorff–Carathéodory fra i valori di funzioni multivoche, analitiche secondo Oka, non è subarmonica.

Let D be a bounded domain of **C**, and let c be the Carathéodory distance on D. According to Theorem I of [5], the function $(x, y) \mapsto \log c(x, y)$ is plurisubharmonic on $D \times D$. Thus, for any domain U of **C** and for all holomorphic maps f, g of U into D, the function $\zeta \mapsto \log c(f(\zeta), g(\zeta))$ is subharmonic on U.

Let f, g be analytic set-valued functions on U, in the sense of K. Oka [2, 3], taking their values in the family of all compact subsets of D. Let h_d and h_c be the Hausdorff distances between subsets of D, defined respectively in terms of the euclidean distance d on **C** and of the Carathéodory distance c on D: for $H^j \subset D$ $(j = 1, 2)$,

$$h_d(H^1, H^2) = \max \{\sup \{d(z, H^2) : z \in H^1\}, \sup \{d(H^1, z) : z \in H^2\}\},$$

$$h_c(H^1, H^2) = \max \{\sup \{c(z, H^2) : z \in H^1\}, \sup \{c(H^1, z) : z \in H^2\}\},$$

where

$$d(z, H^j) = d(H^j, z) = \inf \{|z - u| : u \in H^j\},$$

$$c(z, H^j) = c(H^j, z) = \inf \{c(z, u) : u \in H^j\}, \qquad (j = 1, 2).$$

The question arises whether the functions $\zeta \mapsto h_d(f(\zeta), g(\zeta))$, $\zeta \mapsto h_c(f(\zeta), g(\zeta))$, are subharmonic on U. The present paper will provide a negative answer to this question, contrary to a statement added in proof to [6]. The basic tool will be a result of [3], whereby for any complex Banach algebra A and any holomorphic map $F : U \to A$ the function $f : \zeta \mapsto \mathrm{Sp}\, F(\zeta)$, mapping $\zeta \in U$ onto the spectrum $\mathrm{Sp}\, F(\zeta)$ of $F(\zeta)$, is Oka-analytic. Let F be such that $\mathrm{Sp}\, F(\zeta) \subset D$ for all $\zeta \in U$. Examples will be constructed showing that, for some compact set $K \subset D$, both the functions

$$\zeta \mapsto h_d(\mathrm{Sp}\, F(\zeta), K) \quad , \quad \zeta \mapsto h_c(\mathrm{Sp}\, F(\zeta), K) \tag{1}$$

are not subharmonic on U. This fact entails that Theorem I of [4] —which is a consequence of Theorem I of [5]—does not extend to Oka-analytic functions.

(*) Presentata nella seduta del 12 febbraio 1983.

1. For $s > r > 0$, let

$$C(r, s) = \{z \in \mathbf{C} : r < |z| < s\} \quad , \quad \Delta(r) = \{z \in \mathbf{C} : |z| < r\} .$$

In nn. 1, 2, U will be the unit disc $\Delta = \Delta(1)$. Let $A = l^\infty$, the commutative unital complex Banach algebra of all bounded sequences $x = \{x_n\}$ of complex numbers x_n $(n = 0, 1, \cdots)$ with component-wise addition and multiplication, and with norm $\|x\| = \sup |x_n|$. Let $a = \{a_n\}$ and $b = \{b_n\}$ be dense, respectively, in $\Delta\left(\frac{1}{3}\right)$ and $C\left(\frac{1}{4}, \frac{1}{2}\right)$. Then $\mathrm{Sp}\, b = \overline{C\left(\frac{1}{4}, \frac{1}{2}\right)}$ and, for the holomorphic map $\zeta \mapsto \zeta a$ of Δ into l^∞, $\mathrm{Sp}\, \zeta a = \overline{\Delta\left(\frac{|\zeta|}{3}\right)}$. For any $\zeta \in \Delta$

$$h_d(\mathrm{Sp}\, \zeta a, \mathrm{Sp}\, b) = \max\left(\frac{1}{4}, \frac{1}{2} - \frac{|\zeta|}{3}\right).$$

Thus,

$$h_d(\mathrm{Sp}\, \zeta a, \mathrm{Sp}\, b) = \frac{1}{2} - \frac{|\zeta|}{3} \qquad \text{if } |\zeta| \leq \frac{3}{4},$$

$$h_d(\mathrm{Sp}\, \zeta a, \mathrm{Sp}\, b) = \frac{1}{4} \qquad \text{if } \frac{3}{4} \leq |\zeta| < 1 .$$

Hence the continuous non-constant function $\zeta \mapsto h_d(\mathrm{Sp}\, \zeta a, \mathrm{Sp}\, b)$, which reaches its maximum on Δ at $\zeta = 0$, is not subharmonic on Δ.

2. The above example will now be suitably modified so as to provide an example of a non-subharmonic function defined in terms of h_c.

If the domain D is the unit disc Δ, the Carathéodory distance c coincides with the Poincaré distance

$$\omega(z_1, z_2) = \frac{1}{2} \log \frac{1 + [z_1, z_2]}{1 - [z_1, z_2]},$$

where

$$[z_1, z_2] = \left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right| \quad (z_1, z_2 \in \Delta) .$$

It is well known that the function $(z_1, z_2) \mapsto [z_1, z_2]$ defines a distance on Δ, and it is easily checked that $\omega(z_1, z_2)$ is a strictly increasing function of $[z_1, z_2]$ for z_1 and z_2 in Δ. The geodesic lines through the center 0 of Δ for the Poincaré metric are the radii of the disc Δ. These facts, coupled with the invariance of $\mathrm{Sp}\, b$ under rotations around 0, imply that, for $|z| \leq \frac{1}{4}$, the distance $[z, \mathrm{Sp}\, b] = [\mathrm{Sp}\, b, z] = \inf\{[z, u] : u \in \mathrm{Sp}\, b\}$ between z and $\mathrm{Sp}\, b$ is given by

$$[z, \mathrm{Sp}\, b] = \left[|z|, \frac{1}{4} \right].$$

Therefore, for $\zeta \in \Delta$,

$$\sup\{[z, \mathrm{Sp}\, b] : z \in \mathrm{Sp}\, \zeta a\} = \left[0, \frac{1}{4}\right] = \frac{1}{4}. \tag{2}$$

Similarly, for $\zeta \in \Delta$,

$$\sup\{[z, \mathrm{Sp}\, \zeta a] : z \in \mathrm{Sp}\, b\} = \left[\frac{|\zeta|}{3}, \frac{1}{2}\right] = \frac{3 - 2|\zeta|}{6 - |\zeta|}. \tag{3}$$

Comparison of (2) and (3) shows that

$$h_c(\mathrm{Sp}\, \zeta a, \mathrm{Sp}\, b) = \frac{3 - 2|\zeta|}{6 - |\zeta|} \qquad \text{if} \quad |\zeta| \leq \frac{6}{7},$$

$$h_c(\mathrm{Sp}\, \zeta a, \mathrm{Sp}\, b) = \frac{1}{4} \qquad \text{if} \quad \frac{6}{7} \leq |\zeta| < 1.$$

Hence the continuous, non-constant function $\zeta \mapsto h_c(\mathrm{Sp}\, \zeta a, \mathrm{Sp}\, b)$ on Δ, which reaches its maximum at $\zeta = 0$, is not subharmonic.

3. Examples of holomorphic maps into a non-commutative Banach algebra will now be constructed, for which the functions (1) are not upper semicontinuous.

For any p such that $1 \leq p < \infty$, let $l^p(-\infty, +\infty)$ be the complex Banach space of all bilateral sequences $x = \{x_n\}$ of complex numbers x_n $(n \in \mathbf{Z})$ with the norm

$$\|x\| = (\Sigma |x_n|^p)^{1/p}.$$

Let A be the unital complex Banach algebra of all bounded linear operators on $l^p(-\infty, +\infty)$, and let T, S be the elements of A defined on the canonical basis $\{e_n\}$ of $l^p(-\infty, \infty)$ by

$$\mathrm{T}e_0 = 0, \qquad \mathrm{T}e_n = e_{n-1} \qquad \text{for all} \quad n \neq 0,$$

$$\mathrm{S}e_0 = e_{-1}, \qquad \mathrm{S}e_n = 0 \qquad \text{for all} \quad n \neq 0.$$

Let $\mathrm{F} : \mathbf{C} \to \mathrm{A}$ be defined by $\mathrm{F}(\zeta) = \mathrm{T} + \zeta\mathrm{S}$. Then [1; p. 210] $\mathrm{Sp}\,\mathrm{F}(0) = \bar{\Delta}$, $\mathrm{Sp}\,\mathrm{F}(\zeta) = \partial\Delta = \{z \in \mathbf{C} : |z| = 1\}$ if $\zeta \neq 0$. Therefore

$$h_d(\mathrm{Sp}\,\mathrm{F}(\zeta), \mathrm{Sp}\,\mathrm{T}) = 1 \qquad \text{for all} \quad \zeta \neq 0,$$

$$h_d(\mathrm{Sp}\,\mathrm{F}(0), \mathrm{Sp}\,\mathrm{T}) = 0.$$

That shows that $\zeta \mapsto h_d(\mathrm{Sp}\,F(\zeta), \mathrm{Sp}\,T)$ is not upper semicontinuous. The above example can be adapted to the Poincaré distance on Δ. Choose

$$G(\zeta) = \tfrac{1}{2} F(\zeta).$$

Then

$$h_c(\mathrm{Sp}\,G(\zeta), \mathrm{Sp}\,G(0)) = \omega(0, \tfrac{1}{2})$$

whenever $\zeta \neq 0$, showing that $\zeta \mapsto h_c(\mathrm{Sp}\,G(\zeta), \mathrm{Sp}\,G(0))$ is not upper semicontinuous.

BIBLIOGRAFIA

[1] T. KATO (1966) – *Perturbation theory for linear operators*, Springer–Verlag, Berlin–Heidelberg – New York.

[2] K. OKA (1934) – *Note sur les familles de fonctions analytiques multiformes* etc., « J. Science Hiroshima Univ. », *4*, 93–98.

[3] Z. SLODKOWSKI (1981) – *Analytic set-valued functions and spectra*, « Math. Ann. », *256*, 363–386.

[4] E. VESENTINI (1979) – *Variations on a theme of Carathéodory*, « Ann. Scuola Norm. Sup. Pisa », (4) 7, 39–68.

[5] E. VESENTINI (1982) – *Complex geodesics and holomorphic maps*, « Symposia Mathematica », XXVI, 211–230.

[6] E. VESENTINI (1983) – *Carathéodory distances in Banach algebras*, « Advances in Math. », *47*, 50–73.