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Non-subharmonicity of the Hausdorff distance


<http://www.bdim.eu/item?id=RLINA_1983_8_74_2_62_0>
**Geometria.** — *Non-subharmonicity of the Hausdorff distance.* Nota (*) del Corrisp. EDOARDO VESSENTINI.

**Riassunto.** — Si dimostra con esempi che la distanza di Hausdorff–Carathéodory fra i valori di funzioni multivoche, analitiche secondo Oka, non è subharmonica.

Let $D$ be a bounded domain of $\mathbb{C}$, and let $c$ be the Carathéodory distance on $D$. According to Theorem I of [5], the function $(x, y) \mapsto \log c(x, y)$ is plurisubharmonic on $D \times D$. Thus, for any domain $U$ of $\mathbb{C}$ and for all holomorphic maps $f, g$ of $U$ into $D$, the function $\zeta \mapsto \log c(f(\zeta), g(\zeta))$ is subharmonic on $U$.

Let $f, g$ be analytic set-valued functions on $U$, in the sense of K. Oka [2, 3], taking their values in the family of all compact subsets of $D$. Let $h_d$ and $h_c$ be the Hausdorff distances between subsets of $D$, defined respectively in terms of the euclidean distance $d$ on $\mathbb{C}$ and of the Carathéodory distance $c$ on $D$: for $H^j \subset D (j = 1, 2),$

\[
h_d(H^1, H^2) = \max \{\sup \{d(z, H^2) : z \in H^1\}, \sup \{d(H^1, z) : z \in H^2\}\},
\]

\[
h_c(H^1, H^2) = \max \{\sup \{c(z, H^2) : z \in H^1\}, \sup \{c(H^1, z) : z \in H^2\}\},
\]

where

\[
d(z, H^j) = d(H^j, z) = \inf \{|z - u| : u \in H^j\},
\]

\[
c(z, H^j) = c(H^j, z) = \inf \{c(z, u) : u \in H^j\}, \quad (j = 1, 2).
\]

The question arises whether the functions $\zeta \mapsto h_d(f(\zeta), g(\zeta))$, $\zeta \mapsto h_c(f(\zeta), g(\zeta))$, are subharmonic on $U$. The present paper will provide a negative answer to this question, contrary to a statement added in proof to [6]. The basic tool will be a result of [3], whereby for any complex Banach algebra $A$ and any holomorphic map $F : U \to A$ the function $f : \zeta \mapsto \text{Sp} F(\zeta)$, mapping $\zeta \in U$ onto the spectrum $\text{Sp} F(\zeta)$ of $F(\zeta)$, is Oka-analytic. Let $F$ be such that $\text{Sp} F(\zeta) \subset D$ for all $\zeta \in U$. Examples will be constructed showing that, for some compact set $K \subset D$, both the functions

\[\zeta \mapsto h_d(\text{Sp} F(\zeta), K), \quad \zeta \mapsto h_c(\text{Sp} F(\zeta), K)\]

are not subharmonic on $U$. This fact entails that Theorem I of [4] —which is a consequence of Theorem I of [5]—does not extend to Oka-analytic functions.

(*) Presentata nella seduta del 12 febbraio 1983.
1. For \( s > r > 0 \), let

\[
C(r, s) = \{ z \in \mathbb{C} : r < |z| < s \}, \quad \Delta(r) = \{ z \in \mathbb{C} : |z| < r \}.
\]

In nn. 1, 2, \( U \) will be the unit disc \( \Delta = \Delta(1) \). Let \( \Lambda = l^\infty \), the commutative unital complex Banach algebra of all bounded sequences \( x = \{ x_n \} \) of complex numbers \( x_n (n = 0, 1, \ldots) \) with component-wise addition and multiplication, and with norm \( \|x\| = \sup |x_n| \). Let \( a = \{ a_n \} \) and \( b = \{ b_n \} \) be dense, respectively, in \( \Delta \left( \frac{1}{3} \right) \) and \( C \left( \frac{1}{4}, \frac{1}{2} \right) \). Then \( \text{Sp } b = C \left( \frac{1}{4}, \frac{1}{2} \right) \)

and, for the holomorphic map \( \zeta \mapsto \zeta a \) of \( \Lambda \) into \( l^\infty \), \( \text{Sp } \zeta a = \Delta \left( \frac{|\zeta|}{3} \right) \). For any \( \zeta \in \Delta \)

\[
h_d(\text{Sp } \zeta a, \text{Sp } b) = \max \left( \frac{1}{4}, \frac{1}{2} - \frac{|\zeta|}{3} \right).
\]

Thus,

\[
h_d(\text{Sp } \zeta a, \text{Sp } b) = \begin{cases} 
\frac{1}{2} - \frac{|\zeta|}{3} & \text{if } |\zeta| \leq \frac{3}{4}, \\
\frac{1}{4} & \text{if } \frac{3}{4} \leq |\zeta| < 1.
\end{cases}
\]

Hence the continuous non-constant function \( \zeta \mapsto h_d(\text{Sp } \zeta a, \text{Sp } b) \), which reaches its maximum on \( \Delta \) at \( \zeta = 0 \), is not subharmonic on \( \Delta \).

2. The above example will now be suitably modified so as to provide an example of a non-subharmonic function defined in terms of \( h_e \).

If the domain \( D \) is the unit disc \( \Delta \), the Carathéodory distance \( c \) coincides with the Poincaré distance

\[
\omega (z_1, z_2) = \frac{1}{2} \log \frac{1 + [z_1, z_2]}{1 - [z_1, z_2]},
\]

where

\[
[z_1, z_2] = \left| \frac{z_1 - z_2}{1 - \overline{z_1} z_2} \right| (z_1, z_2 \in \Delta).
\]

It is well known that the function \( (z_1, z_2) \mapsto [z_1, z_2] \) defines a distance on \( \Delta \), and it is easily checked that \( \omega (z_1, z_2) \) is a strictly increasing function of \( [z_1, z_2] \) for \( z_1 \) and \( z_2 \) in \( \Delta \). The geodesic lines through the center 0 of \( \Delta \) for the Poincaré metric are the radii of the disc \( \Delta \). These facts, coupled with the invariance of \( \text{Sp } b \) under rotations around 0, imply that, for \( |z| \leq \frac{1}{4} \),
the distance \([x, \text{Sp } b] = [\text{Sp } b, x] = \inf \{[x, u] : u \in \text{Sp } b\}\) between \(x\) and \(\text{Sp } b\) is given by

\[
[x, \text{Sp } b] = \left[|x|, \frac{1}{4}\right].
\]

Therefore, for \(\zeta \in \Delta\),

\[
(2) \quad \sup \{[x, \text{Sp } b] : x \in \text{Sp } \zeta a\} = \left[0, \frac{1}{4}\right] = \frac{1}{4}.
\]

Similarly, for \(\zeta \in \Delta\),

\[
(3) \quad \sup \{[x, \text{Sp } \zeta a] : x \in \text{Sp } b\} = \left[\frac{|\zeta|}{3}, \frac{1}{2}\right] = \frac{3 - 2|\zeta|}{6 - |\zeta|}.
\]

Comparison of (2) and (3) shows that

\[
h_c(\text{Sp } \zeta a, \text{Sp } b) = \frac{3 - 2|\zeta|}{6 - |\zeta|} \quad \text{if} \quad |\zeta| \leq \frac{6}{7},
\]

\[
h_c(\text{Sp } \zeta a, \text{Sp } b) = \frac{1}{4} \quad \text{if} \quad \frac{6}{7} \leq |\zeta| < 1.
\]

Hence the continuous, non-constant function \(\zeta \mapsto h_c(\text{Sp } \zeta a, \text{Sp } b)\) on \(\Delta\), which reaches its maximum at \(\zeta = 0\), is not subharmonic.

3. Examples of holomorphic maps into a non-commutative Banach algebra will now be constructed, for which the functions (1) are not upper semicontinuous.

For any \(p\) such that \(1 \leq p < \infty\), let \(l^p\) \((\infty, + \infty)\) be the complex Banach space of all bilateral sequences \(x = \{x_n\}\) of complex numbers \(x_n (n \in \mathbb{Z})\) with the norm

\[
\|x\| = \left(\sum |x_n|^p\right)^{1/p}.
\]

Let \(A\) be the unital complex Banach algebra of all bounded linear operators on \(l^p\) \((\infty, + \infty)\), and let \(T, S\) be the elements of \(A\) defined on the canonical basis \(\{e_n\}\) of \(l^p\) \((\infty, \infty)\) by

\[
Te_0 = 0, \quad Te_n = e_{n-1} \quad \text{for all } n \neq 0,
\]

\[
Se_0 = e_{-1}, \quad Se_n = 0 \quad \text{for all } n \neq 0.
\]

Let \(F : C \to A\) be defined by \(F(\zeta) = T + \zeta S\). Then \([1; \text{ p. } 210]\) \(\text{Sp } F(0) = \overline{\Delta}, \text{Sp } F(\zeta) = \partial \Delta = \{z \in C : |z| = 1\}\) if \(\zeta \neq 0\). Therefore

\[
h_d(\text{Sp } F(\zeta), \text{Sp } T) = 1 \quad \text{for all } \zeta \neq 0,
\]

\[
h_d(\text{Sp } F(0), \text{Sp } T) = 0.
\]
That shows that $\zeta \mapsto h_d(\text{Sp } F(\zeta), \text{Sp } T)$ is not upper semicontinuous.

The above example can be adapted to the Poincaré distance on $\Delta$. Choose

$$G(\zeta) = \frac{1}{2} F(\zeta).$$

Then

$$h_c(\text{Sp } G(\zeta), \text{Sp } G(0)) = \omega(0, \frac{1}{2})$$

whenever $\zeta \neq 0$, showing that $\zeta \mapsto h_c(\text{Sp } G(\zeta), \text{Sp } G(0))$ is not upper semicontinuous.

BIBLIOGRAFIA


