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The Lagrangian and Hamiltonian formulations for the waves in an incompressible fluid with the Hall current

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Magnetofluidodinamica. — *The Lagrangian and Hamiltonian formulations for the waves in an incompressible fluid with the Hall current.* Nota di GIULIO MATTEI (*), presentata (**) dal Socio D. GRAFFI.

RIASSUNTO. — In questo lavoro si ricavano: 1) l'equazione d'onda linearizzata, 2) la formulazione Lagrangiana, 3) la formulazione Hamiltoniana, nella teoria della propagazione ondosa in un fluido incomprimibile descritto dalle equazioni della magnetofluidodinamica ideale in presenza di corrente Hall.

1. INTRODUCTION

In this paper we give: (*i*) the linearized wave equation, (*ii*) the Lagrangian formulation, (*iii*) the Hamiltonian formulation, in the theory of wave propagation in a fluid described by the equations of ideal magneto-fluid-dynamics (MFD) in the presence of the Hall current (1). Adopting the terminology introduced in [2], we term this fluid a *Hall current fluid* (HCF).

The fluid is considered incompressible, so that the basic equations are those used, for example, in [3], [4], [5] (2).

2. THE LINEARIZED WAVE EQUATION FOR AN INCOMPRESSIBLE HCF

For an incompressible HCF the basic linearized equations are (Gaussian units)

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} = -\operatorname{grad} p + \frac{1}{4\pi\mu}(\operatorname{curl} \boldsymbol{b}) \wedge \mathbf{B}_0 \tag{2.1}$$

$$\operatorname{div} \boldsymbol{v} = 0 \tag{2.2}$$

$$\frac{\partial \boldsymbol{b}}{\partial t} = \operatorname{curl}(\boldsymbol{v} \wedge \mathbf{B}_0) + \beta \operatorname{curl}(\mathbf{B}_0 \wedge \operatorname{curl} \boldsymbol{b}) \tag{2.3}$$

$$\operatorname{div} \boldsymbol{b} = 0 \,, \tag{2.3'}$$

where ρ is the (constant) density, $\boldsymbol{v}$ is the perturbation in velocity, t is the time, p is the perturbation in pressure, μ is the (constant) magnetic permeability,

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(1) On this subject in MFD in the absence of the Hall effect see [1].
(2) For numerous references about HCF see, for example, [4], [5], [6].

$\boldsymbol{b}/\mu$ is the perturbation in magnetic field, $\mathbf{B}_0/\mu$ is the (constant) magnetic field in the unperturbed (constant) state and β is given by

$$\beta = \frac{c^2 \beta_H}{4 \pi \mu} ; \tag{2.4}$$

in (2.4) c is the speed of light in a vacuum and β_H is the Hall coefficient.

Taking the curl of (2.1) and using (2.3), we obtain

$$\frac{\partial \boldsymbol{q}}{\partial t} = \operatorname{curl} (\boldsymbol{v} \wedge \mathbf{B}_0) \tag{2.5}$$

where

$$\boldsymbol{q} = \boldsymbol{b} + 4 \pi \mu \rho \beta \omega \qquad (\omega = \operatorname{curl} \boldsymbol{v}) . \tag{2.6}$$

In the linear approximation we may write the velocity $\boldsymbol{v}$ in the form

$$\boldsymbol{v} = \frac{\partial \boldsymbol{s}}{\partial t} , \tag{2.7}$$

where $\boldsymbol{s} = \boldsymbol{s} (x_1 , x_2 , x_3 , t)$ is the displacement of a fluid element from its equilibrium position. Substituting in (2.5) and integrating, we obtain

$$\boldsymbol{q} = \operatorname{curl} (\boldsymbol{s} \wedge \mathbf{B}_0) \tag{2.8}$$

so that

$$\boldsymbol{b} = \operatorname{curl} (\boldsymbol{s} \wedge \mathbf{B}_0) - 4 \pi \mu \rho \beta \omega \qquad \left(\omega = \frac{\partial}{\partial t} \operatorname{curl} \boldsymbol{s}\right) . \tag{2.9}$$

Taking the unit vector $\boldsymbol{e}_3$ of the x_3-axis parallel to $\mathbf{B}_0$, we may write (2.9) in the form

$$\boldsymbol{b} = \mathrm{B}_0 \frac{\partial \boldsymbol{s}}{\partial x_3} - 4 \pi \mu \rho \beta \omega . \tag{2.10}$$

Then, following the developments used in [7] n. 4.1 for the case $\beta = 0$ (ordinary MFD without the Hall effect), we obtain

$$\left(\frac{\partial^2}{\partial t^2} - \mathrm{A}_0^2 \frac{\partial^2}{\partial x_3^2} \right) \boldsymbol{s} + \beta \mathrm{B}_0 \frac{\partial \omega}{\partial x_3} = 0 , \tag{2.11}$$

where

$$\mathbf{A}_0 = \frac{\mathbf{B}_0}{\sqrt{4 \mu \pi \rho_0}} \tag{2.12}$$

is the *Alfvén velocity*. Equation (2.11) is the *wave equation*.

If, for example, we are interested in the study of harmonic plane waves propagating along $\mathbf{B}_0$

$$\boldsymbol{s} (x_3 , t) = \tilde{\boldsymbol{s}} \exp [i (2 \pi \nu t - k x_3)] , \tag{2.13}$$

from (2.11), using standard developments, we obtain the dispersion equation

$$(A_0^2 k^2 - 4\pi^2 \nu^2)^2 - (\beta B_0 \, 2\pi\nu k^2)^2 = 0 . \tag{2.14}$$

Eq. (2.14) is studied in [4] and [5], where it is obtained following a way different than that used here.

3. THE LAGRANGIAN FORMULATION FOR CONTINUOUS SYSTEMS

It is well-known (see, for example, [8] Sect 11–2, [9] Sect. 1, [10] Sect 45, [11] p. 118) that a spatially homogeneous continuous system in which energy is conserved can be described in terms of a *Lagrangian density* L, which is a function of (say) n generalized coordinates (field variables) $\psi_h(x_k, t)$, together with their first derivatives

$$\dot{\psi}_h \overset{\text{def}}{=} \frac{\partial \psi_h}{\partial t} \quad \text{and} \quad \psi_{h,k} \overset{\text{def}}{=} \frac{\partial \psi_h}{\partial x_k} \qquad (h = 1, 2, \cdots, n \,;\, k = 1, 2, 3), \tag{3.1}$$

so that

$$L = L(\psi_h, \dot{\psi}_h, \psi_{h,k}) . \tag{3.2}$$

For each ψ_h there will be an equation of motion (field equation) in the form

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\psi}_h}\right) + \sum_{k=1}^{3} \frac{\partial}{\partial x_k}\left(\frac{\partial L}{\partial \psi_{h,k}}\right) - \frac{\partial L}{\partial \psi_h} = 0 \tag{3.3}$$

which can be derived from Hamilton's principle.

4. THE HAMILTONIAN FORMULATION FOR CONTINUOUS SYSTEMS

It is possible (see, for example, [8] Sect. 11–4, [9] Sect. 1, [10] Sect. 45) to give a Hamiltonian formulation for the continuous systems considered in Sect. 3, defining a *canonical momentum density*

$$\pi_h \overset{\text{def}}{=} \frac{\partial L}{\partial \dot{\psi}_h} \tag{4.1}$$

and a *Hamiltonian density*

$$H \overset{\text{def}}{=} \sum_{h=1}^{n} \pi_h \dot{\psi}_h - L . \tag{4.2}$$

The corresponding canonical field equations are

$$\dot{\pi}_h = -\frac{\partial H}{\partial t} + \sum_{k=1}^{3} \frac{\partial}{\partial x_k}\left(\frac{\partial H}{\partial \psi_{h,k}}\right) \quad ; \quad \dot{\psi}_h = \frac{\partial H}{\partial \pi_h} . \tag{4.3}$$

5. The wave equation for an incompressible HCF deduced from a Lagrangian formulation

Our goal is to find the correct Lagrangian density for an incompressible HCF leading to the wave equation (2.11).

At first we identify the field variables ψ_h introduced in Sect. 3 with the components s_h $(h = 1, 2, 3)$ of the vector $\boldsymbol{s}$ introduced in Sect. 2. Then we note the following facts: (*i*) the total energy density w for a HCF is given by (see, for example, [2] and [5]) $w = w_* + B^2/8\pi\mu$, where w_* is the energy density for a non conducting fluid and $\mathbf{B} = \mathbf{B}_0 + \boldsymbol{b}$; (*ii*) the term

$$\frac{\mathbf{B}_0 \cdot \boldsymbol{b}}{4\,\mu\pi} = \rho A_0^2 \frac{\partial s_3}{\partial x_3} - \rho\beta B_0\,\omega_3 \qquad \text{(see (2.10))}$$

cannot contribute (as well as-obviously-the term $B_0^2/8\pi\mu$, which is the constant magnetic energy of the unperturbed state) to the equation of motion (see (3.3)). Then, for (*i*)–(*ii*), we may write the required Lagrangian density in the form

$$L = \tfrac{1}{2}\rho\dot{s}^2 - \frac{b^2}{8\pi\mu},$$

with $\dot{\boldsymbol{s}} \overset{\text{def}}{=} \dfrac{\partial \boldsymbol{s}}{\partial t}$ and $\boldsymbol{b}$ given by (2.10). Noting that the term $\omega^2 = (\operatorname{curl}\dot{\boldsymbol{s}})^2$ cannot contribute to the equation of motion (see (3.3)), we may write the Lagrangian density in the final from

$$L = \frac{\rho}{2}\left[\dot{s}^2 - A_0^2\left(\frac{\partial \boldsymbol{s}}{\partial x_3}\right)^2\right] + B_0\rho\beta\,\frac{\partial \boldsymbol{s}}{\partial x_3}\cdot\boldsymbol{\omega}. \tag{5.1}$$

The Lagrangian density (5.1) does lead exactly to the wave equation (2.11), as it may be readily verified using (3.3).

So we have achieved our goal of describing wave propagation in an incompressible HCF by a Lagrangian formulation.

6. The wave equation for an incompressible HCF deduced from a Hamiltonian formulation

For an incompressible HCF the canonical momentum density (cf. (4.1) and (5.1)) is

$$\pi_h = \rho\,\dot{s}_h \tag{6.1}$$

and the Hamiltonian density is (cf. (4.2), (5.1) and (6.1))

$$H = \frac{1}{2\rho}\sum_{h=1}^{3}\pi_h^2 - \Phi \tag{6.2}$$

where

$$\Phi = -\frac{\rho A_0^2}{2}\left(\frac{\partial s}{\partial x_3}\right)^2 + B_0\,\rho\beta\,\frac{\partial s}{\partial x_3}\cdot\omega\,. \tag{6.3}$$

The corresponding canonical field equations are (cf. (4.3))

$$\dot{s}_h = \pi_h/\rho\,, \tag{6.4}$$

which merely repeats (6.1), and

$$\dot{\pi}_h = -\sum_{k=1}^{3}\frac{\partial}{\partial x_k}\left(\frac{\partial\Phi}{\partial s_{h,k}}\right), \tag{6.5}$$

with Φ given by (6.3).

It may be readily verified that (6.5), using (6.4), does lead to the wave equation (2.11).

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