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A pointwise estimate for the solution to a linear Volterra integral equation

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Fisica matematica. — *A pointwise estimate for the solution to a linear Volterra integral equation* (*). Nota di ANGELO MORRO, presentata (**) dal Socio D. GRAFFI.

RIASSUNTO. — Utilizzando una generalizzazione della disuguaglianza di Gronwall si fornisce una stima puntuale per la soluzione dell'equazione lineare integrale di Volterra di seconda specie. Tale stima può essere applicata utilmente anche nello studio della stabilità di equazioni di evoluzione per mezzi continui.

1. INTRODUCTION

Besides being interesting on their own, Gronwall-like inequalities find significant applications within the theory of differential and integral inequalities (see, e.g., [1–3] and refs. therein). In particular, Gronwall-like inequalities are widely used in proving uniqueness, boundedness, and continuous dependence. Moreover, they are handy yet effective tools as one proceeds to the more delicate and sophisticated considerations in the theory of perturbations and stability.

Among the various generalizations of the Gronwall inequality, Jones' inequality [4], which in turn has been generalized by myself [5], is a simple yet powerful tool. Strangely enough, however, Jones' inequality does not appear to have attracted much attention. Indeed, often the literature is concerned with an estimate established by Willett [6] in spite of Jones' estimate being sharper than Willett's.

It is the purpose of this note to give evidence to the utility of Jones' inequality in searching for pointwise estimate of solutions to linear Volterra integral equations of the second kind [7, 8] which may be written as

$$v(t) = k(t) + \int_0^t \mu(t, s) v(s) \, ds, \qquad t \geq 0, \tag{1.1}$$

where v is the unknown function and k, μ are non-negative known functions. In fact, setting aside the simple case when the kernel μ is separable, first I derive an expression, equivalent to (1.1), involving the partial derivative $\mu_t(t, s)$, besides $\mu(t, t)$. Then, upon assuming that $\mu_t(t, s)$ is separable (in a suitable weak sense), I establish a pointwise estimate of the solution to (1.1).

In ending this section I mention that my interest in the integral equation (1.1) arose in connection with the mathematical properties of materials with

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hidden variables and materials with memory. The estimate obtained in this note is also likely to be very fruitful within the framework of continuum thermodynamics; detailed applications will be dealt with in a future paper.

2. SOME SIMPLE ESTIMATES

Often, when dealing with ordinary differential equations, one is led to examine linear integral inequalities of the form

$$v(t) \leq k(t) + \int_0^t \mu(t,s)\, v(s)\, \mathrm{d}s\,, \qquad t \geq 0\,, \tag{2.1}$$

where v may denote either the solution to the considered differential equation or the perturbation of the solution; k is a known function and the kernel μ maps $\mathrm{R}^+ \times \mathrm{R}^+$ into R^+. The problem then consists in finding a genuine estimate for the unknown function v. As the literature shows — see, e.g., [2] p. 14ff —, this problem has been paid much attention and has been given satisfactory answers. Sharper pointwise estimates of v are possible as long as the kernel is directly separable, namely $\mu(t,s) = g(t)\, h(s)$. One of these estimates is provided by the following lemma whose proof is omitted because it is given in [5].

LEMMA 1. *Let k, g, h be real continuous functions on the interval* $[0, \mathrm{T}]$, $\mathrm{T} > 0$, *and let $gh \geq 0$. If a continuous function v has the property that*

$$v(t) \leq k(t) + g(t) \int_0^t h(s)\, v(s)\, \mathrm{d}s\,, \qquad t \in [0, \mathrm{T}]\,, \tag{2.2}$$

then

$$v(t) \leq k(t) + g(t) \int_0^t h(s)\, k(s) \exp\left[\int_s^t h(\xi)\, g(\xi)\, \mathrm{d}\xi\right] \mathrm{d}s\,, \qquad t \in [0, \mathrm{T}]. \tag{2.3}$$

□

The comparison of this estimate with those of Jones and Willett is immediate. Jones' estimate coincides with (2.3) apart from the stronger assumption $g \geq 0$, $h \geq 0$ made by Jones. Willett's estimate, namely

$$v(t) \leq k(t) + g(t) \int_0^t h(s)\, k(s) \exp\left[\int_0^t h(\xi)\, g(\xi)\, \mathrm{d}\xi\right] \mathrm{d}s\,, \qquad t \in [0, \mathrm{T}]\,,$$

follows from (2.3) provided only that k is non-negative.

3. A POINTWISE ESTIMATE FOR THE SOLUTION TO (1.1)

Look now at the linear Volterra integral equation (1.1). If the kernel μ is differentiable then (1.1) may be written in an equivalent form involving the partial derivative $\mu_t(t, s)$ and $\mu(t, t)$. The new form is convenient whenever simple upper bounds for $\mu_t(t, s)$ and $\mu(t, t)$ are available.

LEMMA 2. *Let the function* $\mu(t, s), t \geq s$, *be differentiable on* $\mathrm{R}^+ \times \mathrm{R}^+$ *and put* $m(t) = \mu(t, t)$. *If* k *and* v *are continuous functions on* R^+ *and* v *meets* (1.1) *then*

$$v(t) = k(t) + \int_0^t m(z)\, k(z) \exp\left[\int_z^t m(\xi)\, d\xi\right] dz + \int_0^t \exp\left[\int_z^t m(\xi)\, d\xi\right] \int_0^z \mu_z(z, s)\, v(s)\, ds\, dz . \tag{3.1}$$

Proof. Let

$$x(t) = \int_0^t \mu(t, s)\, v(s)\, ds .$$

Hence it follows that x is differentiable and the derivative $\dot{x}$ is given by

$$\dot{x}(t) = m(t)\, v(t) + \int_0^t \mu_t(t, s)\, v(s)\, ds .$$

In view of (1.1) we have $v(t) = k(t) + x(t)$ and then

$$\dot{x}(t) - m(t)\, x(t) = m(t)\, k(t) + \int_0^t \mu_t(t, s)\, v(s)\, ds$$

whence

$$\frac{d}{dt}\left\{x(t) \exp\left[-\int_0^t m(\xi)\, d\xi\right]\right\} = m(t)\, k(t) \exp\left[-\int_0^t m(\xi)\, d\xi\right] + \exp\left[-\int_0^t m(\xi)\, d\xi\right] \int_0^t \mu_t(t, s)\, v(s)\, ds .$$

So, since $x(0) = 0$, an obvious integration and substitution of $x(t) = v(t) - k(t)$ yield (3.1). $\square$

On the basis of the expression (3.1) we are now able to get a genuine estimate for the solution v to (1.1) provided that $\mu_t(t, s)$ is separable and $k(t) + \int_0^t m(z)\, k(z) \exp \int_z^t m(\xi)\, d\xi\, dz$ is a positive nondecreasing function; the last feature is ensured by the conditions $m \geq 0$, $k \geq 0$. To arrive at such an estimate we need the following results.

THEOREM 1. *Let f, g, h, n be continuous functions on R^+. Moreover, let f, g, h map R^+ into R^+ and n, mapping R^{++} into R^{++}, be nondecreasing on R^+. Put $\overline{M}(t) = \max[g(t), f(t)\, h(t)]$, $t \geq 0$. If the continuous function $w : R^+ \to R^+$ satisfies*

$$w(t) \leq n(t) + f(t) \int_0^t g(s) \left[\int_0^s h(\xi)\, w(\xi)\, d\xi \right] ds, \qquad t > 0. \tag{3.2}$$

then

$$w(t) \leq n(t) \left\{ 1 + f(t) \int_0^t g(s) \exp \left[\int_s^t \overline{M}(\xi)\, d\xi \right] \int_0^s h(\zeta)\, d\zeta \right\}, \qquad t > 0. \tag{3.3}$$

Proof. Let $t > 0$. Since $n(t) > 0$, on dividing both sides of (3.2) by $n(t)$ and taking advantage of the nondecreasing property of n we may write

$$p(t) \leq 1 + f(t) \int_0^t g(s) \left[\int_0^s h(\xi)\, p(\xi)\, d\xi \right] ds \tag{3.4}$$

where $p(t) = w(t)/n(t)$. The function u defined as

$$u(t) = \int_0^t g(s) \left[\int_0^s h(\xi)\, p(\xi)\, d\xi \right] ds$$

is differentiable and

$$\dot{u}(t) = g(t) \int_0^t h(\xi)\, p(\xi)\, d\xi. \tag{3.5}$$

In terms of u, (3.4) may be written as

$$p(t) \leq 1 + f(t)\, u(t); \tag{3.6}$$

substitution into (3.5) yields

$$\dot{u}(t) \leq g(t)\, H(t) + g(t) \int_0^t h(\xi)\, f(\xi)\, u(\xi)\, d\xi, \tag{3.7}$$

where $H(t) = \int_0^t h(\xi)\,d\xi$. Let

$$v(t) = u(t) + \int_0^t h(\xi)\,f(\xi)\,u(\xi)\,d\xi\,; \tag{3.8}$$

obviously $v(t) \geq u(t)$. Differentiating (3.8) with respect to t delivers

$$\dot{v}(t) = \dot{u}(t) + h(t)\,f(t)\,u(t)\,;$$

substitution of (3.7) gives

$$\dot{v}(t) \leq g(t)\,H(t) + h(t)\,f(t)\,u(t) + g(t)\int_0^t h(\xi)\,f(\xi)\,u(\xi)\,d\xi\,.$$

Hence, in view of the definition (3.8), it follows that

$$\dot{v}(t) \leq g(t)\,H(t) + \overline{M}(t)\,v(t)$$

whence

$$\frac{d}{dt}\left\{v(t)\exp\left[-\int_0^t \overline{M}(\xi)\,d\xi\right]\right\} \leq g(t)\,H(t)\exp\left[-\int_0^t \overline{M}(\xi)\,d\xi\right].$$

Because $v(0) = 0$, on integrating both sides with respect to t and multiplying by $\exp\int_0^t \overline{M}(\xi)\,d\xi$ we have

$$v(t) \leq \int_0^t g(s)\,H(s)\exp\left[\int_s^t \overline{M}(\xi)\,d\xi\right]ds\,.$$

So, (3.6) and the property $v(t) \geq u(t)$ allow us to write

$$p(t) \leq 1 + f(t)\int_0^t g(s)\,H(s)\exp\left[\int_s^t \overline{M}(\xi)\,d\xi\right]ds\,.$$

On multiplying both sides by $n(t)$ we obtain the desired result (3.3). □

We introduce now the main assumption on the kernel $\mu(t, s)$. Specifically, we assume that $\mu_t(t, s)$ is separable in the sense that there exist two continuous functions α, β, mapping R^+ into R^+, such that

$$\mu_t(t, s) \leq \alpha(t)\,\beta(t)\,, \qquad t, s \in R^+.$$

Letting $M(t) = \int_0^t m(\xi)\, d\xi$, we have $\exp\left[\int_s^t m(\xi)\, d\xi\right] \exp[M(t)] \exp[-M(s)]$.

Therefore, if v is a non-negative function, upon the identifications

$$n(t) = k(t) + \exp[M(t)] \int_0^t m(s)\, k(s) \exp[-M(s)]\, ds\,, \qquad k : R^+ \to R^+,$$

$$f(t) = \exp[M(t)\,,$$

$$g(s) = \alpha(s) \exp[-M(s)]\,,$$

$$h(\xi) = \beta(\xi)\,,$$

it follows from (3.1) that (3.2) holds with $w(t) = v(t)$. If n is nondecreasing then, owing to Theorem 1, the result (3.3) provides the desired estimate

$$v(t) \leq \left\{k(t) + \int_0^t m(s)\, k(s) \exp\left[\int_s^t m(\xi)\, d\xi\right] ds\right\} \times \tag{3.9}$$

$$\times \left\{1 + \int_0^t \exp\left[\int_s^t (m + \overline{M})(\xi)\, d\xi\right] \alpha(s) \int_0^s \beta(\xi)\, d\xi\right\}.$$

If however, the values of v need not be non-negative then the assumption

$$|\mu_t(t\,, s)| \leq \alpha(t)\, \beta(s)$$

allows us to arrive at the estimate (3.9) for $|v(t)|$, instead of $v(t)$.

Remark. Sometimes [9] separability is assumed in the form

$$\mu_t(t\,, s) \leq \sum_1^n{}_i\, \alpha_i(t)\, \beta_i(s)\,.$$

If such is the case, the previous results hold unchanged provided we look at $\alpha\beta$ as the inner product in $\mathscr{E}^n$, namely

$$\alpha(s)\, \beta(\xi) = \sum_1^n{}_i\, \alpha_i(s)\, \beta_i(\xi)\,.$$

REFERENCES

[1] V. LAKSHMIKANTHAM and S. LEELA (1969) – *Differential and Integral Inequalities*, Academic Press, New York, vol. 1.

[2] W. WALTER (1970) – *Differential and Integral Inequalities*, Springer–Verlag, Berlin.

[3] C.–C. Yeh (1982) - *Bellman–Bihari integral inequalities in several independent variables*, « J. Math. Anal. Appl. », *87*, 311–321.

[4] G. S. Jones (1964) – *Fundamental inequalities for discrete and discontinuous functional equations*, « J. Soc. Indust. Appl. Math. », *12*, 43–57.

[5] A. Morro (1982) – *A Gronwall-like inequality and its application to continuum thermodynamics*, « Boll. Un. Mat. It. », *B 1*, 553–562.

[6] D. Willett (1965) – *A linear generalization of Gronwall's inequality*, « Prov. Amer. Math. Soc. », *16*, 774–778.

[7] V. Volterra (1896) – *Sulla inversione degli integrali definiti*, « Atti R. Accad. Sci. Torino », *31*; 311, 400, 557, 693.

[8] F. Tricomi (1957) – *Integral Equations*, Interscience, New York.

[9] U. D. Dhongade and S. G. Deo (1974) – *Pointwise estimates of solutions of some Volterra integral equations*, « J. Math. Anal. Appl. », *45*, 615–628.